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Row reduction

$$y_i(c) = \begin{array}{c|cccccccc} & & & & i & i+1 & & & \\ \hline & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 1 & & & & & \\ i & & & & c & 1 & & & \\ i+1 & & & & 0 & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{array}$$

Use

$$\begin{pmatrix} 10 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{10}{7} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

Then

$$\begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix} = y_2\left(\tfrac{1}{8}\right) \begin{pmatrix} 7 & 6 & 2 & 4 \\ 8 & 6 & 3 & 5 \\ 0 & \frac{59}{8} & \frac{53}{8} & \frac{67}{8} \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

$$= y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 6/8 & \frac{-5}{8} & \frac{-3}{8} \\ 0 & \frac{59}{8} & \frac{53}{8} & \frac{67}{8} \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

$$= y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{59}{8}\right) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 6/8 & \frac{-5}{8} & \frac{-3}{8} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & \frac{-5}{8} & \frac{-49}{8} \end{pmatrix}$$

$$= y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{59}{8}\right) y_2\left(\tfrac{6}{8}\right) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & \frac{-11}{8} & -\frac{15}{8} \\ 0 & 0 & \frac{-5}{8} & -\frac{49}{8} \end{pmatrix}$$

$$= y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{59}{8}\right) y_2\left(\tfrac{6}{8}\right) y_3\left(\tfrac{11}{5}\right) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & \frac{-5}{8} & \frac{-49}{8} \\ 0 & 0 & 0 & \frac{464}{40} \end{pmatrix}$$

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Flag variety $n=3$

$$G = GL_3(\mathbb{C}) \quad \text{and} \quad B = \left\{ \begin{pmatrix} b_1 & b_{12} & b_{13} \\ 0 & b_2 & b_{23} \\ 0 & 0 & b_3 \end{pmatrix} \right\} \subseteq GL_3(\mathbb{C})$$

The flag variety is

$$G/B = \{ gB \mid g \in G \}$$

The Weyl group W is the subgroup of G generated by $s_1, \dots, s_{n-1}$, where

$$s_i = y_i(0) = \begin{array}{c|c} & \begin{array}{cc} i & i+1 \end{array} \\ \hline \begin{array}{c} \\ i \\ i+1 \\ \\ \end{array} & \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix} \end{array}, \text{ for } i \in \{1, \dots, n-1\}.$$

So $W = \{1, s_1, s_2, s_1 s_2, s_2 s_1, s_2 s_1 s_2\}$

$$G = \bigsqcup_{w \in W} BwB, \quad \text{and}$$

$$B s_2 s_1 s_2 B = \bigsqcup_{c_1, c_2, c_3 \in \mathbb{C}} y_2(c_1) y_1(c_2) y_2(c_3) B$$

$$B s_1 s_2 B = \bigsqcup_{c_1, c_2 \in \mathbb{C}} y_2(c_1) y_1(c_2) B$$

$$B s_2 s_1 B = \bigsqcup_{c_1, c_2 \in \mathbb{C}} y_1(c_1) y_2(c_2) B$$

$$B s_2 B = \bigsqcup_{c \in \mathbb{C}} y_2(c) B$$

$$B s_1 B = \bigsqcup_{c \in \mathbb{C}} y_1(c) B$$

$$B = B$$

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Cohomology : For $n=3$

$$\begin{tikzcd}
 & Bs_2s_1s_2B \arrow[dl, "c_1\to\infty"'] \arrow[dr, "c_3\to\infty"] & \\
Bs_1s_2B \arrow[d, "c_1\to\infty"'] \arrow[drr, "c_2\to\infty"] & & Bs_2s_1B \arrow[d, "c_1\to\infty"] \arrow[dll, "c_2\to\infty"'] \\
Bs_2B \arrow[dr, "c\to\infty"'] & & Bs_1B \arrow[dl, "c\to\infty"] \\
 & B &
\end{tikzcd}$$

$\mathbb{C}^3$ $\quad H^0(G/B)=\overline{\mathbb{Q}}_\ell$

$\mathbb{C}^2$ $\quad \mathbb{C}^2$ $\quad H^1(G/B)=\overline{\mathbb{Q}}_\ell\oplus\overline{\mathbb{Q}}_\ell$

$\mathbb{C}$ $\quad \mathbb{C}$ $\quad H^2(G/B)=\overline{\mathbb{Q}}_\ell\oplus\overline{\mathbb{Q}}_\ell$

pt $\quad H^3(G/B)=\overline{\mathbb{Q}}_\ell$

Grothendieck, Sém. C. Chevalley (1958) p 1-36, Prop. 7

Structure For $n=2$,

$W=\{1,s_1\}$ and $G/B = Bs_1B \sqcup B$ with

$Bs_1B = \bigsqcup_{c\in\mathbb{C}} y_1(c)B$ and $B=B$.

So

$G/B \cong \mathbb{C}\sqcup pt =$

$\sqcup\ B$

then

$$\lim_{c\to 0} y_1(c)B = y_1(0)B = s_1B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}B$$

$$\lim_{c\to\infty} y_1(c)B = \lim_{c\to\infty}\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}B = \lim_{c^{-1}\to 0}\begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix}\begin{pmatrix} c & 1 \\ 0 & -c^{-1} \end{pmatrix}B$$

$$= \lim_{c^{-1}\to 0}\begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix}B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}B = B$$

So

$\overline{Bs_1B} = Bs_1B \sqcup B =$ s_1B

B $=$ s_1B

$B = \mathbb{P}^1$!

Bruhat order

If $w = s_{i_1} \cdots s_{i_\ell}$ then

$$\overline{BwB} = \bigsqcup_{v \le w} BvB, \quad \text{where}$$

$v \le w$ if $v \in \{ s_{i_{k_1}} \cdots s_{i_{k_r}} \mid (k_1, \dots, k_r) \text{ is a subsequence of } (1, \dots, \ell) \}$

Diagonal matrices

Let $a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in GL_n(\mathbb{C})$

$$a y_i(c) = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & 1 & \\ & & 1 & 0 & \\ & & & & 1 \ddots 1 \end{pmatrix} = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_{i-1} & & & \\ & & & a_i c & a_i & \\ & & & a_{i+1} & 0 & \\ & & & & & a_{i+2} \ddots a_n \end{pmatrix}$$

$$y_i(a_i a_{i+1}^{-1} c) \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_{i-1} & & & \\ & & & a_{i+1} & & \\ & & & & a_i & \\ & & & & & a_{i+2} \ddots a_n \end{pmatrix} = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_{i-1} & & & \\ & & & a_i c & a_i & \\ & & & a_{i+1} & 0 & \\ & & & & & a_{i+2} \ddots a_n \end{pmatrix} = a y_i(c).$$

So

$$a y_i(c) = y_i(a_i a_{i+1}^{-1} c)(s_i a s_i).$$

T-action on G/B

$$T = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in GL_n(\mathbb{C}) \right\}$$

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T-fixed points

The T-fixed points on G/B are the elements of

$$(G/B)^T = \{yB \mid \text{if } a\in T \text{ then } ayB = yB\}.$$

then

$$(G/B)^T = \coprod_{w\in W} (BwB)^T \quad \text{and}$$

$$(Bs_2s_1s_2B)^T = \{y_2(0)y_1(0)y_2(0)B\} \simeq pt$$

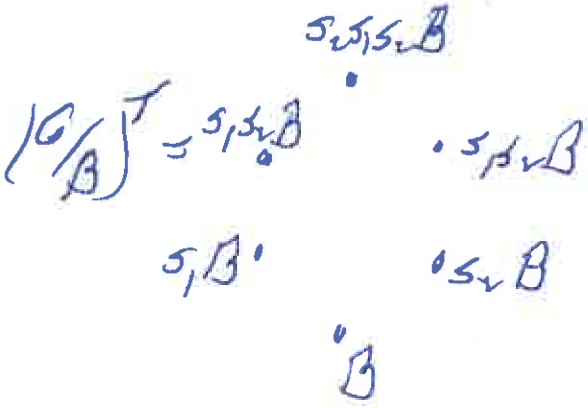
$$(Bs_2s_1B)^T = \{y_2(0)y_1(0)B\} \simeq pt$$

$$(Bs_1s_2B)^T = \{y_1(0)y_2(0)B\} \simeq pt$$

$$(Bs_2B)^T = \{y_2(0)B\} \simeq pt$$

$$(Bs_1B)^T = \{y_1(0)B\} \simeq pt$$

$$B^T = B \simeq pt$$



So $(G/B)^T = \{wB \mid w\in W\}$ and

$$\begin{aligned} W &\longrightarrow (G/B)^T \\ w &\longmapsto wB \end{aligned}$$

is a bijection.

The oracle

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & 0 & & & & & * \\ 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & & \end{pmatrix} \right\} \subseteq M_n(\mathbb{C})$$

(Dyck path)

Let $g \in GL_n(\mathbb{C})$. The <u>Hessenberg variety</u> for the pair $(\mathfrak{m}, g)$ is

$$Y_{\mathfrak{m}}^{-1}(g) = \{ yB \in G/B \mid y^{-1} g y \in \mathfrak{m} \}$$

Let $w \in W$. The <u>Lusztig variety</u> for the pair (w, g) is

$$Y_w^{-1}(g) = \{ yB \in G/B \mid y^{-1} g y \in BwB \}$$

Suppose $\mathbb{C} = \mathbb{F}_7$ (the finite field with 7 elements)

$\mathrm{Card}(Y_{\mathfrak{m}}^{-1}(g)) = ?$ $\qquad$ $\mathrm{Card}(Y_w^{-1}(g)) = ?$

$\mathrm{Card}(Y_{\mathfrak{m}}^{-1}(g) \cap BvB) = ?$ $\qquad$ $\mathrm{Card}(Y_w^{-1}(g) \cap BvB) = ?$