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Murphy elements and Casimirs

Group algebra of S_k: $\mathbb{C}S_k$

Murphys $M_j = \sum_{i=1}^{j-1} s_{ij}$

s_{ij} is transposition switching i and j.

$d = M_1 + \cdots + M_k = \sum_{1 \le i < j \le k} s_{ij}$ is in $Z(\mathbb{C}S_k)$.

The enveloping algebra $U\mathfrak{gl}_n$

Generators: E_{ij} for $i,j \in \{1,\ldots,n\}$.

Relations: $E_{ij}E_{k\ell} = E_{k\ell}E_{ij} + \delta_{jk}E_{i\ell} - \delta_{\ell i}E_{kj}$

Casimir: $\sum_{i,j=1}^{n} E_{ij}E_{ji}$ is in $Z(U\mathfrak{gl}_n)$.

The module $V^{\otimes k}$

$U\mathfrak{gl}_n$ acts by

$E_{ij}(v_1 \otimes \cdots \otimes v_k) = \sum_{\ell=1}^{k} v_1 \otimes \cdots \otimes v_{\ell-1} \otimes E_{ij}v_\ell \otimes v_{\ell+1} \otimes \cdots \otimes v_k.$

$\mathbb{C}S_k$ acts by

$w(v_1 \otimes \cdots \otimes v_k) = v_{w^{-1}(1)} \otimes \cdots \otimes v_{w^{-1}(k)}$

If $V = \mathbb{C}^n$ then, as operators on $V^{\otimes k}$,

$\kappa = d$.

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Transvections and Hecke algebras Rep.Thy Sem

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Group algebra of $GL_n(\mathbb{F}_q)$: $\mathbb{C}G$

$\mathcal{C}$ the conjugacy class of $\begin{pmatrix} 1 & 1 & & \\ 0 & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

A transvection is $x \in \mathcal{C}$

$$C = \sum_{x \in \mathcal{C}} x \quad \text{is in } Z(\mathbb{C}G)$$

The Hecke algebra H: $G = \bigsqcup_{w \in S_n} BwB$.

$H = \mathrm{span}\{T_w \mid w \in S_n\}$ where

$$T_w = \frac{1}{|B|} \sum_{x \in BwB} x, \quad \text{with } B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subseteq G.$$

The module $\mathbb{1}_B^G$: For $g \in G$ let

$$v_g = \frac{1}{|B|} \sum_{x \in gB} x, \quad \text{and } \mathbb{1}_B^G = \mathrm{span}\{v_g \mid g \in G\}.$$

$\mathbb{C}G$ acts by left multiplication on $\mathbb{1}_B^G$

H acts by right multiplication.

Let

$$D = (q-1) \sum_{1 \le i < j \le n} q^{n-1-(j-i)} T_{s_{ij}}.$$

Then $D \in Z(H)$ and, as operators on $\mathbb{1}_B^G$,

$$C = D.$$

Central elements in H

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For $\mu_1, \ldots, \mu_\ell \in \mathbb{Z}_{>0}$ with $\mu_1 + \cdots + \mu_\ell = n$.

$\mathcal{C}_\mu$ is the conj. class of

$$u_\mu = \begin{pmatrix} \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ & \ddots & & \vdots \\ & & & 1 \end{bmatrix}}_{\mu_1} & & \\ & \ddots & \\ & & \overbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ & \ddots & & \vdots \\ & & & 1 \end{bmatrix}}^{\mu_\ell} \end{pmatrix} \text{ in } G$$

$C_\mu = \sum_{x \in \mathcal{C}_\mu} x$ is in $Z(\mathbb{C}G)$.

Find $D_\mu \in Z(H)$ so that, as operators on $\mathbb{1}_B^G$

$$C_\mu = D_\mu$$

<u>Answer:</u>

$$G = \bigsqcup_{\substack{\text{conj} \\ \text{classes}}} \mathcal{C}_\gamma \quad \text{and} \quad G = \bigsqcup_{w \in W} BwB.$$

Let

$$D_{\gamma, w} = \mathrm{Card}(\mathcal{C}_\gamma \cap BwB)$$

and

$$D_\gamma = \sum_{w \in W} D_{\gamma, w}\, q^{-\ell(w)} T_w.$$

Then D_γ is in $Z(H)$ and, as operators on $\mathbb{1}_B^G$,

$$C_\gamma = D_\gamma.$$

Bitraces and Lusztig varieties

As (G,H) bimodules

$$\mathcal{H}_B^G \cong \bigoplus_{\lambda \in \hat{H}} G^\lambda \otimes H^\lambda$$

with

G^λ a simple G-module

H^λ a simple H-module.

$$\chi_G^\lambda : G \longrightarrow \mathbb{C}$$
$$g \longmapsto Tr(g, G^\lambda)$$
is character of G^λ

$$\chi_H^\lambda : H \longrightarrow \mathbb{C}$$
$$h \longmapsto Tr(h, H^\lambda)$$
is character of H^λ.

Then

$$\begin{aligned} D_{\mu,w} &= Card(\mathcal{C}_\mu \cap BwB) \\ &= \frac{|\mathcal{C}_\mu|}{|G/B|} tr(u_\mu T_{w^{-1}}, \mathcal{H}_B^G) \\ &= \frac{|\mathcal{C}_\mu|}{|G/B|} \sum_{\lambda \in \hat{H}} \chi_G^\lambda(u_\mu) \chi_H^\lambda(T_{w^{-1}}) \\ &= \frac{|\mathcal{C}_\mu|}{|G/B|} Card(\mathcal{Y}_{w^{-1}}^{-1}(u_\mu)), \end{aligned}$$

where

$$\mathcal{Y}_{w^{-1}}^{-1}(u_\mu) = \{ yB \in G/B \mid y^{-1} u_\mu y \in Bw^{-1}B \}$$

is the <u>Lusztig variety</u> for the pair (u_μ, w^{-1}).

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<u>Central elements in H</u>

$\mathcal{W}$ an index set for conjugacy classes in W

γ_ν minimal length in the conj. class W_ν.

Two bases of $Z(H)$:

$\{K_\nu^H \mid \nu \in \mathcal{W}\}$ and $\{z_\lambda^H \mid \lambda \in \hat{H}\}$

Geck-Rouquier basis. minimal idempotent basis.

Let $\mathcal{C}_\mu$ be the conj. class of u_μ in G.

$$A_{\mu w} = \frac{|G/B|}{|\mathcal{C}_\mu|} \mathrm{Card}(\mathcal{C}_\mu \cap BwB) = \mathrm{Tr}(u_\mu T_{w^{-1}}, 1_B^G)$$

In $Z(H)$

$$A_\mu = \sum_{w \in W} A_{\mu w} q^{-\ell(w)} T_w$$

$$= \sum_{\nu \in \mathcal{W}} A_{\mu\nu} K_\nu$$

$$= \sum_{\lambda \in \hat{H}} \chi_G^\lambda(u_\nu) z_\lambda^H$$

and

$$z_\lambda^H = \sum_{\nu \in \mathcal{W}} \chi_H^\lambda(T_{\gamma_\nu^{-1}}) K_\nu^H$$

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Symmetric functions

$J_\mu(x_1,\dots x_n;q,t)$ integral form Macdonald polynomials

$S_\lambda(x_1,\dots x_n;t)$ Big Schurs

$m_\nu(x_1,\dots x_n)$ monomial symmetric functions

$$S_\lambda(x;t) = \sum_\nu L_{\lambda\nu}(t)\,(1-t)^{\ell(\nu)}\,m_\nu(x)$$

$$J_\mu(x;q,t) = \sum_\lambda K_{\lambda\mu}(q,t)\,S_\lambda(x;t)$$

$$J_\mu(x;q,t) = \sum_\nu a_{\mu\nu}(q,t)(1-t)^{\ell(\nu)}\,m_\nu(x)$$

Let

$$\tilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)}K_{\lambda\mu}(q,t^{-1}), \quad \tilde{a}_{\mu\nu}(q,t) = t^{n(\mu)}a_{\mu\nu}(q,t$$

Then

$$\mathcal{K}^\lambda_G(u_\mu) = \tilde{K}_{\lambda\mu}(0,q^{-1})$$

$$\mathcal{X}^\lambda_H(T_{\gamma_\nu^{-1}}) = q^n L_{\lambda\nu}(q^{-1})$$

$$Tr(uT_{\gamma_\nu^{-1}}, 1_B^G) = q^n\,\tilde{a}_{\mu\nu}(0,q^{-1}).$$

The matrix $a_{\mu\nu}(q,t)$ is TRIANGULAR!

The point

$$J_\mu(x;q,t) = \sum_\nu a_{\mu\nu}(q,t)\, m_\nu(x)$$

counts points in an affine Lusztig variety.

$$\begin{tikzcd}
LG = G(\mathbb{F}_q((\varepsilon))) \\
\cup\!| \\
K = G(\mathbb{F}_q[[\varepsilon]]) \arrow[rr, "\varepsilon=0"] & & G(\mathbb{F}_q) \\
\cup\!| & & \cup\!| \\
I_\pi \arrow[rr, "\varepsilon=0"] & & P_\pi(\mathbb{F}_q) = \left\{ \begin{pmatrix} \square & & * \\ & \ddots & \\ 0 & & \square \end{pmatrix} \right\} \\
\cup\!| & & \cup\!| \\
I \arrow[rr, "\varepsilon=0"] & & B(\mathbb{F}_q) = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}
\end{tikzcd}$$

The affine Lusztig variety is

$$Y^{-1}_{\gamma_\nu^{-1}}(u_\mu) = \{ yI \in G/I \mid y^{-1}u_\mu y \in I\gamma_\nu I \}$$

A parabolic affine Springer fiber is

$$Y^{-1}_{I_\pi}(u_\mu) = \{ yI_\pi \in G/I_\pi \mid y^{-1}u_\mu y \in I_\pi \}$$
$$= \bigsqcup_{w\in W_\pi} \left(I_\pi/I \times Y_w^{-1}(u_\mu) \right)$$

and this reduces the claim to Mellit's theorem about $\tilde{H}_\mu[X;q,t]$.

The motivation

Macdonald polynomials $P_\lambda(x;q,t)$ $\leftarrow\!\!\!\times\!\!\!\rightsquigarrow$ Modified and Integral Macdonald polynomials $\tilde{H}_\mu[X;q,t]$, $J_\mu(x;q,t)$

except in type GL_n

"because"
$\mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]^W \cong \mathbb{C}[\mathfrak{h}^*]^W$ $\leftarrow\!\!\!\times\!\!\!\rightsquigarrow$ Representations of W

BUT

Center of $Z(H)$. Hecke algebra $\xleftarrow{\text{decategorification}}$ Representations of W

For C_μ a unipotent class in G define

$$A_\mu = \sum_\nu a_{\mu\nu}(q,t)\, K_\nu^H$$

counts points in an affine Lusztig variety.

Does A_μ deserve to be called a Modified Macdonald polynomial?