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Rep. Thy. Seminar ①

# Hessenberg varieties and combinatorics

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$$\begin{array}{ccc} G & & \mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{n}^+ \\ \cup\!| & \text{with} & \cup\!| \\ B & \text{Lie} & \mathfrak{b} = \mathfrak{a} \oplus \mathfrak{n}^+ \\ \cup\!| & \text{algebras} & \cup\!| \\ T & & \mathfrak{a} \end{array} \qquad \begin{array}{l} \mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha \\ \\ \mathfrak{n}^- = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{-\alpha}. \end{array}$$

Let $V$ be a $G$-set and $m$ a $B$-subset of $V$.

$$G \times_B m = \frac{G \times m}{\langle (gb, m) = (g, bm) \rangle} \quad \text{and}$$

the <u>$Gm$-resolution</u> is

$$\begin{array}{c} G \times_B m \\ {\scriptstyle Y_m} \downarrow \\ Gm \subseteq V. \end{array}$$

Let $v \in V$. The <u>Hessenberg variety for $m$ over $v$</u>, or <u>$Y_m$-fiber over $v$</u> is

$$Y_m^{-1}(v) = \{ gB \in G/B \mid g^{-1}v \in m \}.$$

<u>Favourite example</u>: The adjoint representation

$G$ acts on $\mathfrak{g}$, $m$ is a $\mathfrak{b}$-submodule of $\mathfrak{g}$

$$\begin{array}{c} G \times_B m \\ {\scriptstyle Y_m} \downarrow \\ Gm \subseteq \mathfrak{g}. \end{array}$$

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<u>Remark</u> If $v \notin G\mathfrak{n}$ then $\gamma_{\mathfrak{n}}^{-1}(v) = \emptyset$.

<u>Remark</u> $\gamma_{\mathfrak{n}}^{-1}(v) = \bigsqcup_{w \in W} (\gamma_{\mathfrak{n}}^{-1}(v) \cap BwB)$

<u>Remark</u> If $\mathfrak{m} = \mathfrak{b}$ then $\mathfrak{m}^{\perp} = \mathfrak{n}^{+}$,

$G\mathfrak{n}^{+} = \mathcal{N}$ is the <u>nilpotent cone</u>

$$\begin{array}{c} G \times_B \mathfrak{n}^{+} \\ {\scriptstyle \gamma_{\mathfrak{n}^+}} \big\downarrow \\ G\mathfrak{n}^{+} = \mathcal{N} \end{array}$$ is the <u>Springer resolution</u>

$$\begin{array}{c} G \times_B \mathfrak{b} \\ {\scriptstyle \gamma_{\mathfrak{b}}} \big\downarrow \\ \mathfrak{g} \end{array}$$ is the <u>Grothendieck-Springer simultaneous resolution</u>

<u>Remark</u> $G = Sp_{2n}(\mathbb{C})$ and $V = L(\varepsilon_1) \oplus L(\varepsilon_1) \oplus L(\varepsilon_1 + \varepsilon_2)$ and

$\mathfrak{n} = \bigoplus_{\lambda \in Q_{3,0} R^{+} - \{0\}} V_\lambda$ then $G\mathfrak{n} = \mathcal{N}^{\text{exotic}}$

is the <u>exotic nilpotent cone</u>

(see the work of Syu Kato and others).

## Steinberg varieties

The Steinberg variety for $\mathfrak{m}$ is

$$Z_{\mathfrak{m}} = (G \times_B \mathfrak{m}) \times_{G\mathfrak{m}} (G \times_B \mathfrak{m})$$

So

$$Z_{\mathfrak{m}} = \{ (y_1 B, y_2 B, m) \in G/B \times G/B \times \mathfrak{m} \mid m \in y_1 \mathfrak{m} \cap y_2 \mathfrak{m} \}$$

Define a $G \times \mathbb{C}^\times$ actions on

$G \times_B \mathfrak{m}$ by $(g, q)(y, m) = (gy, q^{-1} m)$

$G\mathfrak{m}$ by $(g, q)(x) = q^{-1} g x$

$Z_{\mathfrak{m}}$ by $(g, q)(y_1 B, y_2 B, m) = (g y_1 B, g y_2 B, q^{-1} m)$

Then $Y_{\mathfrak{m}}$ is $G \times \mathbb{C}^\times$ equivariant.

Question: Is $K_{G \times \mathbb{C}^\times}(Z_{\mathfrak{m}})$ an affine Hecke algebra?

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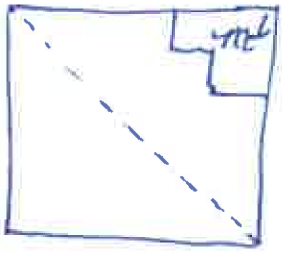
## Comparing $Y_{\mathfrak{m}}$ and $Y_{\mathfrak{m}^\perp}$


and




$$(Y_{\mathfrak{m}^\perp})_* (\mathbb{C}_{Y_{\mathfrak{m}^\perp}}[N_{\mathfrak{m}^\perp}]) \cong \bigoplus_{(n_\lambda,\rho)} IC(\overline{G n_\lambda}, \rho) \otimes V^{\mathfrak{m}^\perp}_{(n_\lambda,\rho)}$$

$$(Y_{\mathfrak{m}})_* (\mathbb{C}_{Y_{\mathfrak{m}}}[N_{\mathfrak{m}}]) \cong \bigoplus_{(n_\lambda,\rho)} IC(\mathfrak{g}, \mathcal{M}_{(n_\lambda,\rho)}) \otimes V^{\mathfrak{m}^\perp}_{(n_\lambda,\rho)} \otimes \mathrm{sgn}$$

where $V^{\mathfrak{m}^\perp}_{(n_\lambda,\rho)}$ is a $\mathbb{Z}$-graded vector space

$\mathcal{M}_{(n_\lambda,\rho)}$ is a local system supported on $\mathfrak{g}^{rs}$ corresponding to an irred. rep. $W^{(n_\lambda,\rho)}$ of $W$

$[N_{\mathfrak{m}^\perp}]$ and $[N_{\mathfrak{m}}]$ are appropriate shifts.

## Fourier transform

$$\mathcal{F}((Y_{\mathfrak{m}^\perp})_* (\mathbb{C}_{Y_{\mathfrak{m}^\perp}}[N_{\mathfrak{m}^\perp}])) = (Y_{\mathfrak{m}})_* (\mathbb{C}_{Y_{\mathfrak{m}}}[N_{\mathfrak{m}}])$$

$$\mathcal{F}(IC(\overline{G n_\lambda}, \rho)) = IC(\mathfrak{g}, \mathcal{M}_{(n_\lambda,\rho)})$$

## Support

$$V^{\mathfrak{m}^\perp}_{(n_\lambda,\rho)} \neq 0 \Rightarrow V^{\mathfrak{m}^\perp}_{(n_\lambda,\rho)} \neq 0.$$

$H_T(\mathcal{Y}_m^{-1}(s))$ for $s$ regular semisimple

Define $R_m^-$ by

$$m = \Big(\bigoplus_{-\alpha \in R_m^-} \mathfrak{g}_{-\alpha}\Big) \oplus \mathfrak{b}, \qquad m^\perp = \Big(\bigoplus_{-\alpha \in R^- - R_m^-} \mathfrak{g}_\alpha\Big)$$

$$\mathfrak{g} = G m$$

Let $s \in \mathfrak{g}$ be regular semisimple.

$T$ acts on $\mathcal{Y}_m^{-1}(s)$

The moment graph of $\mathcal{Y}_m^{-1}(s)$ has

Vertices: $\{T\text{-fixed points on } \mathcal{Y}_m^{-1}(s)\}$
$= \{wB \mid w \in W\}$

Edges: $\{1\text{-dim'l } T\text{-orbits on } \mathcal{Y}_m^{-1}(s)\}$
$= \{wB \xleftrightarrow{\alpha} s_\alpha wB \mid w^{-1}\alpha \in R_m^-\}$.

Let

$$\bigoplus_{w \in W} H_T(pt) = \{(f_w)_{w \in W} \mid f_w \in H_T(pt)\}$$

with pointwise product $(fg)_w = f_w g_w$

Then $H_T(\mathcal{Y}_m^{-1}(s))$ is the subalgebra

$$H_T(\mathcal{Y}_m^{-1}(s)) = \left\{ (f_w)_{w \in W} \;\middle|\; \begin{array}{l} \text{if } w^{-1}\alpha \in R_m^- \text{ then} \\ f_w - f_{ws_\alpha} \in \alpha H_T(pt) \end{array} \right\}$$

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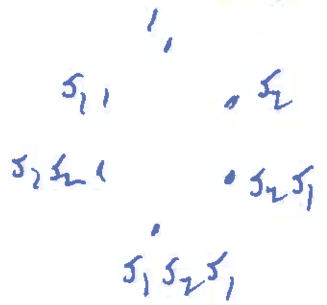
## Examples for $\mathfrak{g} = \mathfrak{gl}_3$

$m = \mathfrak{b} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$

$m^\perp = \mathfrak{n}^+ = \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$

moment graph of $\mathcal{Y}_m^{-1}(s)$




$R_m^- = \emptyset =$ 1 2 3

$X_m = 6m_{1^3} + 3m_{21} + m_3$

$= s_{(3)} + 2s_{(21)} + s_{(1^3)}$

$= e_1^3$

$G_m = 6m_{1^3} + 3m_{21} + m_3$

$= s_{(3)} + 2s_{(21)} + s_{(1^3)}$

$= e_1^3$

with

$s_{1^3} = m_{1^3} = e_3$

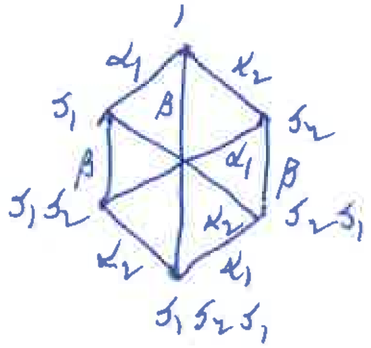
$s_{21} = m_{21} + 2m_{1^3}$

$s_3 = m_3 + m_{21} + m_{1^3}$

$m = \mathfrak{g} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$

$m^\perp = 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$R_m^- = \{-\alpha_1, -\alpha_2, -\beta\} =$


$X_m = (1 + 2q + 2q^2 + q^3)\, m_{1^3}$

$= [3]!\, e_3 = [3]!\, s_{(1^3)}$

$G_m = (1 + 2q + 2q^2 + q^3)\, m_{1^3}$

$+ (1 + q + q^2)\, m_{21}$

$+ m_3$

$\cancel{= (1+q^2)(s_{(21)} + q s_{(1^3)})}$

$= s_{(3)} + (q + q^2)\, s_{(21)} + q^3 s_{(1^3)}$

## Nil affine Hecke algebra

The algebra

$$S = H_T(pt) = \mathbb{Z}[y_1, \ldots, y_n]$$

is generated by $y_\lambda$, $\lambda \in \mathfrak{a}^*_{\mathbb{Z}}$, with

$$y_{\lambda+\mu} = y_\lambda + y_\mu .$$

The <u>nil affine Hecke algebra</u>

$$H = S \otimes S \otimes \mathbb{C}W = \mathrm{span}\left\{ f(x) g(y) t_w \;\middle|\; \begin{matrix} f, g \in S, \\ w \in W \end{matrix} \right\}$$

where $x_\lambda = 1 \otimes y_\lambda$ and $y_\lambda = y_\lambda \otimes 1$ and

$$t_w x_\lambda = x_\lambda t_w \quad \text{and} \quad t_w y_\lambda = y_{w\lambda} t_w .$$

In $\bigoplus_{w \in W} H_T(pt)$ let $b_v = (b_{vw})_{w \in W}$ with $b_{vw} = \delta_{vw}$.
Then $1 = \sum_{w \in W} b_w$ and $t_v b_w = b_{vw}$

$$x_\lambda \cdot 1 = \sum_{w \in W} y_{w^{-1}\lambda} b_w , \qquad y_\lambda \cdot 1 = \sum_{w \in W} y_\lambda b_w$$

defines an action of $H$ on $\bigoplus_{w \in W} H_T(pt)$

and on

$$H_T(Y_m^{-1}(s)).$$

For $\mathfrak{g} = \mathfrak{gl}_3$,

$$x_\lambda = \begin{array}{ccc} & y_\lambda & \\ y_{s_1\lambda} & & y_{s_2\lambda} \\ y_{s_2 s_1\lambda} & & y_{s_1 s_2 \lambda} \\ & y_{s_1 s_2 s_1 \lambda} & \end{array} \qquad \text{and} \qquad y_\lambda = \begin{array}{ccc} & y_\lambda & \\ y_\lambda & & y_\lambda \\ y_\lambda & & y_\lambda \\ & y_\lambda & \end{array}$$

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# Chromatic symmetric functions and LLT polynomials

$$\mathfrak{m} = \Big(\bigoplus_{\alpha\in R_{\mathfrak{m}}^-} \mathfrak{g}_\alpha\Big) \oplus \mathfrak{h}$$

Assume $\mathfrak{g} = \mathfrak{gl}_n$. View $R_{\mathfrak{m}}^-$ as a graph with

Vertices: $\{1, 2, \ldots, n\}$

Edges: $\underset{i}{\bullet}\!\!-\!\!\underset{j}{\bullet}$ if $\varepsilon_j - \varepsilon_i \in R_{\mathfrak{m}}^-$

A coloring of $R_{\mathfrak{m}}^-$ is a function

$$\kappa : \{1, \ldots, n\} \to \mathbb{Z}_{>0}$$

Define

$$\mathrm{asc}_{\mathfrak{m}}(\kappa) = \#\{\varepsilon_j - \varepsilon_i \in R_{\mathfrak{m}}^- \mid \kappa(\varepsilon_j - \varepsilon_i) > 0\}$$

A proper coloring of $R_{\mathfrak{m}}^-$ is a coloring $\kappa$ such that

if $\varepsilon_j - \varepsilon_i \in R_{\mathfrak{m}}^-$ then $\kappa(\varepsilon_j - \varepsilon_i) \neq 0$.

$$X_{\mathfrak{m}} = \sum_{\substack{\text{proper}\\ \text{colorings}\\ \text{of } R_{\mathfrak{m}}^-}} t^{\mathrm{asc}_{\mathfrak{m}}(\kappa)} x_{\kappa(1)} \cdots x_{\kappa(n)}$$

chromatic symmetric function of $R_{\mathfrak{m}}^-$

$$G_{\mathfrak{m}} = \sum_{\substack{\text{colorings}\\ \text{of } R_{\mathfrak{m}}^-}} q^{\mathrm{asc}_{\mathfrak{m}}(\kappa)} x_{\kappa(1)} \cdots x_{\kappa(n)}$$

(unicellular) LLT polynomial of $R_{\mathfrak{m}}^-$

Remark These are degree $n$ and correspond to representations of $S_n$

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# Cohomology representations and LLT representations

Let
$$\varepsilon : H_T(pt) \longrightarrow H(pt)$$
be the map that forgets the $T$-action.

The cohomology representation of $Y_m^{-1}(s)$ is the $S \ltimes \mathbb{C}W$-module given by
$$H^*(Y_m^{-1}(s)) = \frac{H_T^*(Y_m^{-1}(s))}{\langle g(y)-\varepsilon(g) \mid g \in S\rangle}$$
quotient by the ideal gen by $\{g(y)-\varepsilon(g) \mid g\in S\}$.

The LLT-representation of $Y_m^{-1}(s)$ is the $S \otimes \mathbb{C}W$-module given by
$$LLT(Y_m^{-1}(s)) = \frac{H_T^*(Y_m^{-1}(s))}{\langle f(x)-\varepsilon(f) \mid f \in S\rangle}.$$

Theorem Let $W^\lambda$ be the simple $W$-module indexed by $\lambda$. Then
$$X_m = \sum_\lambda \sum_j [H^j(Y_m^{-1}(s)) : W^\lambda]\, q^j s_\lambda$$
$$G_m = \sum_\lambda \sum_j [LLT^j(Y_m^{-1}(s)) : W^\lambda]\, q^j s_\lambda$$
where $s_\lambda$ is the Schur function indexed by $\lambda$.

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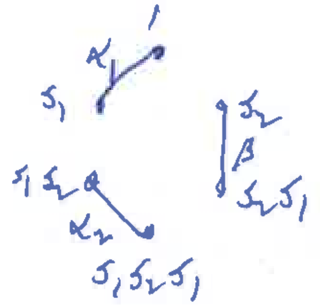
# Examples continued for $y = \mathfrak{gl}_3$

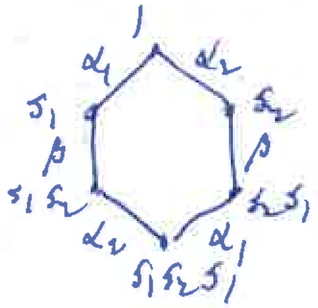
$$\mathfrak{m} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \qquad \mathfrak{m} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix}$$

$$\mathfrak{m}^\perp = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathfrak{m}^\perp = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$R_{\mathfrak{m}}^- = \{-\alpha_1\} = $ 1 — 2   3 $\qquad R_{\mathfrak{m}}^- = \{-\alpha_1, -\alpha_2\} = $ 1 — 2 — 3

moment graph of $Y_{\mathfrak{m}}^{-1}(s)$




$$X_{\mathfrak{m}} = 3(1+q)m_{1^3} + (1+q)m_{21} = (1+q)s_{(2,1)} + (1+q)s_{(1,1,1)}$$

$$X_{\mathfrak{m}} = (1+4q+q^2)m_{1^3} + q\,m_{21} = (1+2q+q^2)s_{(1,1,1)} + q\,s_{(2,1)}$$

$$G_{\mathfrak{m}} = 3(1+q)m_{1^3} + (2+q)m_{21} + m_3 = (s_{(2)} + q\,s_{(1,1)})s_{(1)} = s_{(3)} + (1+q)s_{(2,1)} + q\,s_{(1,1,1)}$$

$$G_{\mathfrak{m}} = (1+4q+q^2)m_{1^3} + (1+2q)m_{21} + m_3 = s_{(3)} + 2q\,s_{(2,1)} + q^2 s_{(1,1,1)}$$

<u>Note:</u> $H^*(Y_P^{-1}(s)) = P(P/B)\,\mathrm{Ind}_{W_P}^W(\mathbb{1})$

$LLT_q(Y_P^{-1}(s)) = \mathrm{Ind}_{W_P}^W(C_P)$, where $C_P$ is the coinvariant ring.