17.01.2024 ①
Macdonald
n line

__1 line__ Let $p = r + si \in \mathbb{C}^\times$.

The line is

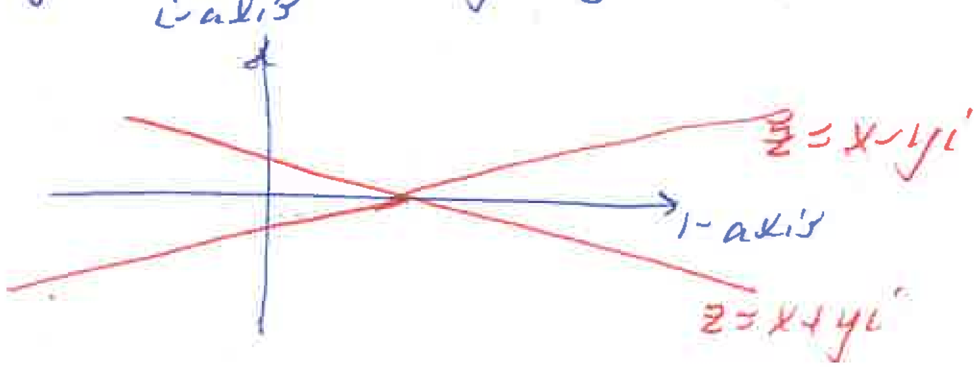
$$1 = \{ z \in \mathbb{C} \mid \frac{z}{p} + \frac{\bar z}{\bar p} = 1 \} = \{ z \in \mathbb{C} \mid \bar z = \frac{-\bar p}{p}(z - p) \}$$

Writing $z = x + yi$ then $\bar p z + p \bar z = p \bar p$ gives

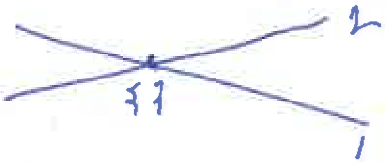
$$rx + sy + (ry - sx)i + rx + sy + (sx - ry)i = r^2 + s^2$$

so that

$$2(rx + sy) = r^2 + s^2 \text{ and } y = \frac{-r}{s}x + \tfrac{1}{2}(r^2 + s^2).$$



__2-lines__



Let $p_1, p_2 \in \mathbb{C}^\times$

$$1 = \{ z \in \mathbb{C} \mid \bar z = t_1(z - p_1) \} \quad \text{with } t_1 = \frac{-\bar p_1}{p_1}$$

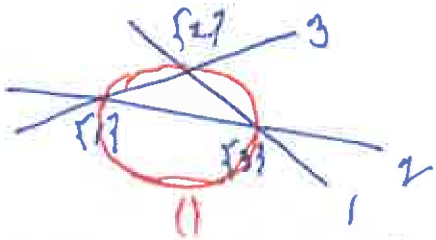
$$2 = \{ z \in \mathbb{C} \mid \bar z = t_2(z - p_2) \} \quad \text{with } t_2 = \frac{-\bar p_2}{p_2}$$

Then

$$\{\ \} = \frac{p_1 t_1}{t_1 - t_2} + \frac{p_2 t_2}{t_2 - t_1}$$

is the point of intersection of the two lines.

3 lines



17.01.2023 ②

Let $p_1, p_2, p_3 \in \mathbb{C}^*$ Macdonald n-line

$$1 = \{ z \in \mathbb{C} \mid \bar{z} = t_1 (z - p_1) \} \quad \text{with } t_1 = -\frac{\bar{p}_1}{p_1}$$

$$2 = \{ z \in \mathbb{C} \mid \bar{z} = t_2 (z - p_2) \} \quad \text{with } t_2 = -\frac{\bar{p}_2}{p_2}$$

$$3 = \{ z \in \mathbb{C} \mid \bar{z} = t_3 (z - p_3) \} \quad \text{with } t_3 = -\frac{\bar{p}_3}{p_3}$$

Let

$$s_1 = t_1 + t_2 + t_3$$

$$c_0 = \frac{p_1 t_1^2}{(t_1 - t_2)(t_1 - t_3)} + \frac{p_2 t_2^2}{(t_2 - t_1)(t_2 - t_3)} + \frac{p_3 t_3^2}{(t_3 - t_1)(t_3 - t_2)}$$

$$c_1 = \frac{p_1 t_1}{(t_1 - t_2)(t_1 - t_3)} + \frac{p_2 t_2}{(t_2 - t_1)(t_2 - t_3)} + \frac{p_3 t_3}{(t_3 - t_1)(t_3 - t_2)}$$

Then

$$() = \{ x = c_0 - c_1 \tau \mid \tau \in \mathbb{C}, |\tau| = 1 \}$$

is a circle containing {1}, {2} and {3} and

c_0 is the circumcentre

$c_0 - c_1 s_1$ is the orthocentre

$c_0 - \frac{1}{2} c_1 s_1$ is the nine-point centre

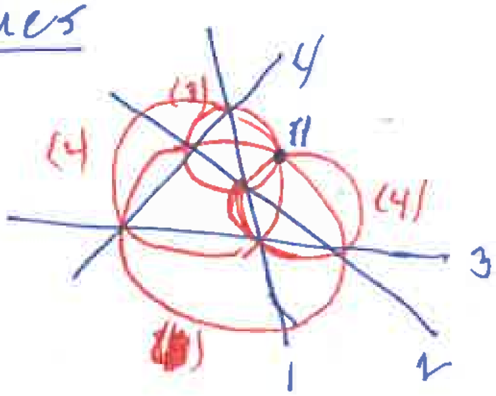
$c_0 - \frac{1}{3} c_1 s_1$ is the centroid

$\{ c_0 - c_1 s_1 p \mid p \in \mathbb{R} \}$ is the Euler line

<u>4 lines</u>

17.01.2023

Macdonald n-line (3)



Let $p_1, p_2, p_3, p_4 \in \mathbb{C}^\times$

$$t_1 = \frac{-\bar{p}_1}{p_1},\quad t_2 = \frac{-\bar{p}_2}{p_2},\quad t_3 = \frac{-\bar{p}_3}{p_3},\quad t_4 = \frac{-\bar{p}_4}{p_4}.$$

Let

$$s_1 = t_1 + t_2 + t_3 + t_4$$

$$s_2 = t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4$$

$$s_3 = t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4$$

$$s_4 = t_1t_2t_3t_4$$

and

$$c_0 = \frac{p_1t_1^3}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} + \frac{p_2t_2^3}{(t_2-t_1)(t_2-t_3)(t_2-t_4)} + \frac{p_3t_3^3}{(t_3-t_1)(t_3-t_2)(t_3-t_4)} + \frac{p_4t_4^3}{(t_4-t_1)(t_4-t_2)(t_4-t_3)}$$

$$c_1 = \frac{p_1t_1^2}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} + \frac{p_2t_2^2}{(t_2-t_1)(t_2-t_3)(t_2-t_4)} + \frac{p_3t_3^2}{(t_3-t_1)(t_3-t_2)(t_3-t_4)} + \frac{p_4t_4^2}{(t_4-t_1)(t_4-t_2)(t_4-t_3)}$$

$$c_2 = \frac{p_1t_1}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} + \frac{p_2t_2}{(t_2-t_1)(t_2-t_3)(t_2-t_4)} + \frac{p_3t_3}{(t_3-t_1)(t_3-t_2)(t_3-t_4)} + \frac{p_4t_4}{(t_4-t_1)(t_4-t_2)(t_4-t_3)}$$

17 ~~11~~.01.2024 (3)
Macdonald

<u>4-line</u>

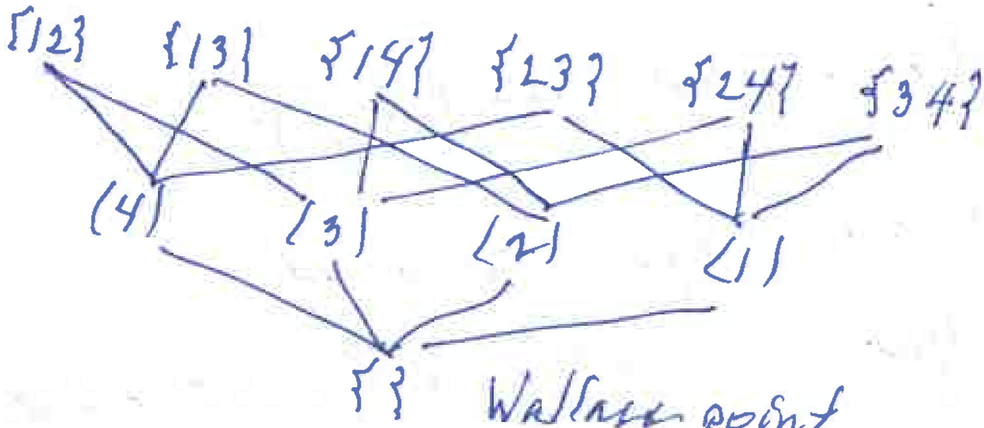
$H = \{ x = c_0 - 2c_1\tau + c_2\tau^2 \mid \tau \in \mathbb{C}, |\tau| = 1 \}$ <u>cardiod</u>

$\{\} = W = c_0 - \frac{c_1^2}{c_2}$ <u>Wallace point</u>

$K = \{ x = c_0 - c_1\tau \mid \tau \in \mathbb{C}, |\tau| = 1 \}$ <u>centric circle</u>

$D = \{ x = c_0 - c_1 s_1 + 2c_1\tau + c_2 s_4 \frac{1}{\tau^2} \mid \tau \in \mathbb{C}, |\tau| = 1 \}$ <u>deltoid</u>

$M = \{ x = c_0 - \frac{1}{2} c_1 s_1 + \frac{1}{2} c_2 (\tau + \frac{s_4}{\tau}) \mid \tau \in \mathbb{C}, |\tau| = 1 \}$ <u>medial line</u>



Wallace point.

K contains the centres of (1), (2), (3), (4)
$\{\} = W$ is contained in (1), (2), (3) and (4)
$\{\} = W$ is the cusp of H
H is tangent to (1), (2), (3) and (4)
D is tangent to 1, 2, 3 and 4 with centre
$c_0 - c_1 s_1$. (orthocentre of the 4-line).

17.01.2024 (5)

Macdonald n-line

<u>n-lines</u> Let $p_1, \dots, p_n \in \mathbb{C}^2$ and

$$t_1 = \frac{-\bar{p}_1}{p_1},\ t_2 = \frac{-\bar{p}_2}{p_2}, \dots, t_n = \frac{-\bar{p}_n}{p_n}$$

Define

$$c_0 = \frac{p_1 t_1^{n-1}}{f_1} + \frac{p_2 t_2^{n-1}}{f_2} + \dots + \frac{p_n t_n^{n-1}}{f_n}$$

$$c_1 = \frac{p_1 t_1^{n-2}}{f_1} + \frac{p_2 t_2^{n-2}}{f_2} + \dots + \frac{p_n t_n^{n-2}}{f_n}$$

$$\vdots$$

$$c_{n-1} = \frac{p_1}{f_1} + \frac{p_2}{f_2} + \dots + \frac{p_n}{f_n}$$

where

$$f_j(t_1, \dots, t_n) = \prod_{\substack{i \in \{1,\dots,n\} \\ i \neq j}} (t_j - t_i)$$

For

<u>3 lines</u> $(1) = \{c_0 - c_1 z \mid |z| = 1, z \in \mathbb{C}\}$ contains $\{1\}, \{2\}, \{3\}$

<u>4 lines</u> $(1) = \{c_0 - c_1 z \mid z \in \mathbb{C}, |z| = 1\}$ contains the centres of $(1), (2), (3), (4)$

<u>5 lines</u> $(1) = \{c_0 - c_1 z \mid z \in \mathbb{C}, |z| = 1$ contains the centres of ~~(12), (13), (14), (23), (24)~~ $(1), (2), (3), (4), (5)$

<u>n-lines</u> have n-circles $(1), (2), \dots (n)$ and

$$\{x = c_0 - 2c_1 z + c_2 z^2 \mid z \in \mathbb{C}, |z| = 1\}$$

is a limaçon tangent to the circles $(1), (2), \dots (n)$.

and $c_0 - \frac{c_1 c_2}{2}$ is point on "the centre circles of $(n-1)$ out of n lines".

Let n be even 17.01.2024 (6)

Macdonald n-line

The <u>Clifford point</u> of the n-line is

$$\begin{vmatrix} c_2 & c_3 \\ c_3 & c_4 \end{vmatrix} x = \begin{vmatrix} c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix}$$

Let n be odd

The <u>Clifford circle</u> of the n-line is

$$\left\{ \begin{vmatrix} c_2 & c_3 \\ c_3 & c_4 \end{vmatrix} x = \begin{vmatrix} c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix} - \theta \begin{vmatrix} c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \\ c_3 & c_4 & c_5 \end{vmatrix} \;\middle|\; \theta \in \mathbb{C},\ |\theta| = 1 \right\}.$$

<u>~~Definition~~ Clifford's chain</u>

Let $1, \dots, n$ be n lines.

<u>Case 1</u> n is odd

For $i \in \{1, \dots, n\}$ there is a Clifford circle (i) for $i^c = \{1, \dots, i-1, i+1, \dots, n\}$.

The <u>Clifford point</u> of $\{l_1, \dots, l_n\}$ is W contained in $(1), \dots, (n)$.

<u>Case 2</u> n is even.

For $i \in \{1, \dots, n\}$ there is a Clifford point $\{i\}$ for $\{l_1, \dots, l_{i-1}, l_{i+1}, \dots, l_n\}$.

The <u>Clifford circle</u> of $\{l_1, \dots, l_n\}$ is the circle M containing $\{1\}, \dots, \{n\}$.

17.01.2024
Maulwalt (7)
n-lines

Longchamps's chain

Let $l_1, \ldots, l_n$ be n lines

For $i \in \{1, \ldots, n\}$ there is a centric circle

C_i for $\{l_1, \ldots, l_{i-1}, l_{i+1}, \ldots, l_n\}$ with centre c_i.

The centric circle of $\{l_1, \ldots, l_n\}$ is

the circle C containing $c_1, \ldots, c_n$.