Macdonald Symmetric Products

Let X be a curve. A <u>curve</u> is

(a) a compact connected Riemann surface of genus g

or, equivalently,

(b) a complete nonsingular algebraic curve over $\mathbb{C}$.

<u>The cohomology of X</u>: $H^*(X;\mathbb{Z})$.

$$H^*(X,\mathbb{Z}) = H^0(X;\mathbb{Z}) \oplus H^1(X;\mathbb{Z}) \oplus H^2(X;\mathbb{Z})$$

with

$H^0(X,\mathbb{Z})$ has $\mathbb{Z}$-basis $\{1\}$

$H^1(X;\mathbb{Z})$ has $\mathbb{Z}$-basis $\{\alpha_1,\dots,\alpha_{2g}\}$

$H^2(X;\mathbb{Z})$ has $\mathbb{Z}$-basis $\{\beta\}$

with relations:

(a) If $j,i \in \{1,\dots,2g\}$ and $j \notin \{i+g, i-g\}$ then

$$\alpha_i \alpha_j = 0$$

(b) If $i \in \{1,\dots,g\}$ then $\alpha_i \alpha_{i+g} = -\alpha_{i+g}\alpha_i = \beta$.

It follows that if $i \in \{1,\dots,2g\}$ then

$$\alpha_i \beta = \beta \alpha_i = 0 \quad \text{and} \quad \beta^2 = 0.$$

Macdonald Symm. Prod.

Cohomology of X^n. Let $\mathbb{F}$ be a field

$$X^n = \underbrace{X \times X \times \cdots \times X}_{n \text{ factors}}$$

Then $H^*(X^n, \mathbb{F}) = H^*(X, \mathbb{F})^{\otimes n}$ is generated by

$$\left\{\alpha_{ik} \;\middle|\; \begin{array}{l} i \in \{1,..,2g\} \\ k \in \{1,..,n\} \end{array}\right\} \cup \{\beta_k \mid k \in \{1,..,n\}\}.$$

with

$$\deg(\alpha_{ik}) = 1, \qquad \deg(\beta_k) = 2$$

and relations

$$\alpha_{ik}\alpha_{jk} = 0, \quad \text{if} \quad i,j \in \{1,..,2g\},\ j \notin \{i+g, i-g\},\ k \in \{1,..,n\}$$

$$\alpha_{ik}\alpha_{i+g,k} = -\alpha_{i+g\,k}\alpha_{ik} = \beta_k, \quad \text{if} \quad i \in \{1,\dots,g\},\ k \in \{1,..\ n\}$$

$$\alpha_{ik}\alpha_{jl} = -\alpha_{jl}\alpha_{ik}, \quad \text{if} \quad i,j \in \{1,..,2g\} \text{ and } k,l \in \{1,\dots,n\}.$$

Then

$$\alpha_{ik}\beta_{k} = \beta_k\alpha_{ik} = 0 \quad \text{and} \quad \beta_k^2 = 0,$$

if $k \in \{1,..,n\}$ and $i \in \{1,\dots,2g\}$.

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Macdonald Symm. Product.

Cohomology of $X(n)$

$$X(n) = X^n/S_n,$$

where $w(x_1,\dots,x_n) = (x_{w^{-1}(1)},\dots,x_{w^{-1}(n)})$

for $w \in S_n$ and $x_1,\dots,x_n \in X$.

Then S_n acts on $H^*(X^n;\mathbb{Z})$ by

$$w(\alpha_{ik}) = \alpha_{i,w^{-1}(k)} \quad \text{and} \quad w(\beta_k) = \beta_{w^{-1}(k)}$$

for $w \in S_n$, $i \in \{1,\dots,2g\}$ and $k \in \{1,\dots,n\}$.

Let

$$f : X^n \longrightarrow X(n) = X^n/S_n$$
$$(x_1,\dots,x_n) \longmapsto [x_1,\dots,x_n].$$

Then

$$H^*(X(n);\mathbb{Z}) \xrightarrow{f^*} H^*(X^n;\mathbb{Z}) \text{ has}$$

$\operatorname{im}(f^*) = H^*(X^n;\mathbb{Z})^{S_n}$ and $\ker(f^*) = 0$.

Then $H^*(X(n),\mathbb{Z})^{S_n}$ has generators

$$\xi_i = \alpha_{i1} + \dots + \alpha_{in}, \quad \text{for } i \in \{1,\dots,2g\}$$

$$\eta = \beta_1 + \dots + \beta_n.$$

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Presentation of $H^*(X(n), \mathbb{Z})$

Let

$$\xi_i' = \xi_{i+g}, \text{ for } i \in \{1, \dots, g\}.$$

Then $H^*(X(n), \mathbb{Z})$ is generated by

$$\xi_1, \dots, \xi_g, \quad \xi_1', \dots, \xi_g', \quad \eta$$

with

$$\deg(\xi_i) = \deg(\xi_i') = 1 \text{ and } \deg(\eta) = 2$$

for $i \in \{1, \dots, g\}$ and relations

(a) if $i, j \in \{1, \dots, g\}$ then

$$\xi_i \xi_j = -\xi_j \xi_i, \quad \xi_i' \xi_j' = -\xi_j' \xi_i', \quad \xi_i \xi_j' = -\xi_j' \xi_i,$$
$$\xi_i \eta = \eta \xi_i, \quad \xi_i' \eta = \eta \xi_i'$$

(b) If $a, b, c, q \in \mathbb{Z}_{\geq 0}$ and $a + b + 2c + q = n + 1$ and $i_1, \dots, i_a, j_1, \dots, j_b, k_1, \dots, k_c$ are distinct elements of $\{1, \dots, g\}$ then

$$\xi_{i_1} \cdots \xi_{i_a} \xi_{j_1}' \cdots \xi_{j_b}' (\xi_{k_1} \xi_{k_1}' - \eta) \cdots (\xi_{k_c} \xi_{k_c}' - \eta) \eta^q = 0.$$

Theorem

$H^*(X(n), \mathbb{Z})$ has no torsion (as a $\mathbb{Z}$-module)

More precisely,

$$H^*(X\langle n\rangle, \mathbb{Z}) = H^0(X\langle n\rangle, \mathbb{Z}) \oplus \cdots \oplus H^{2n}(X\langle n\rangle, \mathbb{Z})$$

and

if $r \in \{0, \ldots, n\}$ then

$H^r(X\langle n\rangle, \mathbb{Z})$ has $\mathbb{Z}$-basis

$$\left\{ \xi_{i_1} \cdots \xi_{i_p} \eta^q \;\middle|\; \begin{array}{l} p \in \{0, \ldots, r\} \text{ and } r = p + 2q \\ i_1, \ldots, i_p \in \{1, \ldots, 2g\} \text{ with } i_1 < \cdots < i_p \end{array} \right\}$$

and

$H^{2n-r}(X\langle n\rangle, \mathbb{Z})$ has $\mathbb{Z}$-basis

$$\left\{ \xi_{i_1} \cdots \xi_{i_p} \eta^q \;\middle|\; \begin{array}{l} p \in \{0, \ldots, r\} \text{ and } 2n - r = p + 2q \\ i_1, \ldots, i_p \in \{1, \ldots, 2g\} \text{ with } i_1 < \cdots < i_p \end{array} \right\}$$

Favourite invariants

$\chi_y =$ coeff of x^n in $(1+xy)^{g-1}(1-x)^{g-1}$ is the $\underline{\chi_y\text{-genus of } X(n)}$

$\chi_{-1} = (-1)^n\binom{2g-2}{n}$ is the $\underline{\text{Euler characteristic of } X(n)}$

$\chi_1 = \begin{cases} (-1)^m\binom{g-1}{m}, & \text{if } n = 2m, \\ 0, & \text{if } n \text{ is odd,} \end{cases}$ is the $\underline{\text{index of } X(n)}$

$\chi_0 = (-1)^n\binom{g-1}{n}$ is the $\underline{\text{Hirzebruch arithmetic genus of } X(n)}$

$p_a = \binom{g-1}{n} + (-1)^{n-1}$ is the $\underline{\text{classical arithmetic genus of } X(n)}$

$p_g = \binom{g}{n}$ is the $\underline{\text{geometric genus of } X(n)}$

$h^{p,q} = \binom{g}{p}\binom{g}{q} + \binom{g}{p-1}\binom{g}{q-1} + \ldots$ are the $\underline{\text{Hodge numbers for } X(n)}$

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The Zeta function

Let X be a curve and

$$Z(t) = \frac{\prod_{i=1}^{2g}(1-p_i t)}{(1-t)(1-qt)}$$ its zeta function

so that

$$\frac{d}{dt}\log Z(t) = \sum_{s\in\mathbb{Z}_{>0}} \mathrm{Card}(X(\mathbb{F}_{q^s}))\, t^{s-1}.$$

Let

$$\Phi_0(t) = (1-t),$$

$$\Phi_k(t) = \prod_{1\le i_1<\cdots<i_k\le 2g} (1-p_{i_1}\cdots p_{i_k} t),$$

for $k\in\{1,\ldots,2g\}$. Let

$$F_k(t) = \begin{cases} \Phi_k(t)\,\Phi_{k-2}(t)\,\Phi_{k-4}(t)\cdots, & \text{if } k\in\{1,\ldots,n\} \\ F_{2n-k}(q^{k-n}t), & \text{if } k\in\{n+1,\ldots,2n\} \end{cases}$$

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The zeta function of $X(n)$ is

$$Z_n(t) = \frac{F_1(t)F_3(t)\cdots F_{2n-1}(t)}{F_0(t)F_2(t)\cdots F_{2n}(t)}.$$

Riemann hypothesis for $X(n)$

All roots of $Z_n(t)$ have absolute value in

$$\{q^{-\frac{1}{2}\cdot 0}, q^{-\frac{1}{2}\cdot 1}, q^{-\frac{1}{2}\cdot 2}, \dots, q^{-\frac{1}{2}\cdot 2n}\}$$

Riemann hypothesis for X

All roots of $Z(t)$ have absolute value in

$$\{q^{-\frac{1}{2}}\}$$

Functional equation for X

$$Z\left(\frac{1}{qt}\right) = (q^{\frac{1}{2}}t)^{2-2g} Z(t)$$

Functional equation for $X(n)$

$$Z_n\left(\frac{1}{q^n t}\right) = (-q^{\frac{1}{2}n}t)^{(-1)^n\binom{2g-2}{n}} Z_n(t)$$