Divided differences and Monk Rules

02.01.2024 ①

Monk rules
Schubert Seminar
A. Ram

S_n acts on $\mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$ by

$$(s_i f)(x_1,\dots,x_n) = f(x_1,\dots,x_{i-1},x_{i+1},x_i,x_{i+2},\dots,x_n).$$

Let

$$\partial_i = (1+s_i)\frac{1}{x_i - x_{i+1}}. \quad \text{Then} \quad \partial_i^2 = 0$$
$$\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}$$

Let

$$D_i = (1+s_i)\frac{1}{1-x_i^{-1}x_{i+1}}. \quad \text{Then} \quad D_i^2 = D_i$$
$$D_iD_{i+1}D_i = D_{i+1}D_iD_{i+1}.$$

Let $t^{\frac12}\in\mathbb{C}^\times$ and

$$c_{s_i} = \frac{t^{-\frac12} - t^{\frac12}x_i^{-1}x_{i+1}}{1-x_i^{-1}x_{i+1}}. \quad \text{Then} \quad c_{s_i}^2 = (t^{\frac12}+t^{-\frac12})c_{s_i}$$
$$c_{s_i}c_{s_{i+1}}c_{s_i} + c_{s_{i+1}}$$
$$= c_{s_{i+1}}c_{s_i}c_{s_{i+1}} + c_{s_i}.$$

Let $T_i = c_{s_i} + t^{-\frac12}$

Then

$$T_i^2 = (t^{\frac12} - t^{-\frac12})T_i + 1$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}.$$

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Monk Rules
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DAHA Let $q\in\mathbb{C}^\times$ and

$$(y_i f)(x_1,\dots,x_n) = f(x_1,\dots,x_{i-1},q^{-1}x_i,x_{i+1},\dots,x_n)$$

Let

$$T_\pi = s_1\cdots s_{n-1}y_n \quad\text{and}\quad y_1 = T_\pi T_{n-1}\cdots T_1$$

and

$$y_i = T_{i-1}^{-1}\, y_{i-1}\, T_{i-1}^{-1}, \quad\text{for } i\in\{2,\dots,n\}.$$

Let

$$\boxed{\tau_i^\vee = c_{s_i} - \frac{t^{-\frac12}-t^{\frac12}y_i^{-1}y_{i+1}}{1-y_i^{-1}y_{i+1}}}$$

Then

$$(\tau_i^\vee)^2 = \frac{(t^{\frac12}-t^{-\frac12}y_iy_{i+1}^{-1})(t^{\frac12}-t^{-\frac12}y_i^{-1}y_{i+1})}{(1-y_iy_{i+1}^{-1})(1-y_i^{-1}y_{i+1})}$$

and

$$\tau_i^\vee\tau_{i+1}^\vee\tau_i^\vee = \tau_{i+1}^\vee\tau_i^\vee\tau_{i+1}^\vee.$$

Let

$$\boxed{\tau_\pi^\vee = X_1T_1\cdots T_{n-1}}$$

where

X_j is the operator on $\mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$

given by

$$(X_jf)(x_1,\dots,x_n) = x_j f(x_1,\dots,x_n).$$

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Monk Rules
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Schubert

$\{\mathfrak{S}_w \mid w \in S_\infty\}$ is a basis for $\mathbb{C}[x_1, x_2, \ldots]$.

(a) If $w(i) > w(i+1)$ then $\partial_i \mathfrak{S}_w = \mathfrak{S}_{ws_i}$.

(b) $\mathfrak{S}_{w_0^{(n)}} = x_1^{n-1} x_2^{n-2} \cdots x_{n-2}^{2} x_{n-1}$

Macdonald

$\{E_\mu \mid \mu \in \mathbb{Z}^n\}$ is a basis of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

(a) If $\mu_i > \mu_{i+1}$ then $t^{\frac{1}{2}} t_i^\vee E_\mu = E_{s_i \mu}$

(b) $E_{(0,\ldots,0)} = 1$

(c) $t_\pi^\vee E_\mu = t^{\text{stuff}} E_{\pi\mu}$, where

$\pi(\mu_1, \ldots, \mu_n) = (\mu_n + 1, \mu_1, \mu_2, \ldots, \mu_{n-1})$.

$\text{STUFF} = \frac{1}{2}(n-1) - \#\{i \mid \mu_i > \mu_{i+1}\}$

(d) $Y_i^\vee E_\mu = q^{-\mu_i} t^{-(v_\mu(i)-1) + \frac{1}{2}(n-1)} E_\mu$

where $v_\mu \in S_n$ is minimal length such that $v_\mu \mu$ is increasing.

Monk Rules: Schubert

(a) $x_1, x_2, \ldots$ generate $\mathbb{C}[x_1, x_2, \ldots]$.

(b) $\mathbb{C}[x_1, x_2, \ldots]$ has basis $\{\mathfrak{S}_w \mid w \in S_\infty\}$.

Compute $x_j \cdot \mathfrak{S}_w$.

Monk rules: Macdonald

(a) $x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}$ generate $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

(b) $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ has basis $\{E_\mu \mid \mu \in \mathbb{Z}^n\}$.

Compute $x_j \cdot E_\mu$ and $x_j^{-1} E_\mu$

Let X_j be the operator on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

$$(X_j f)(x_1, \ldots, x_n) = x_j \cdot f(x_1, \ldots, x_n).$$

Theorem

$$X_j = \sum_{\substack{c \subseteq \{1, \ldots, n\} \\ c \cap \{j\} \neq \phi}} \tau_{c,j}^\vee F_{c,j}(Y) f_c(Y)$$

Corollary

$$x_j E_\mu = \sum_{\substack{c \subseteq \{1, \ldots, n\} \\ c \cap \{j\} \neq \phi}} F_\mu(c, j) \, wt_\mu(c) \, E_{rot_c(\mu)}.$$

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Monk rules
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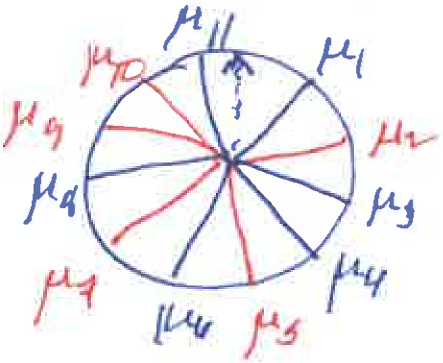
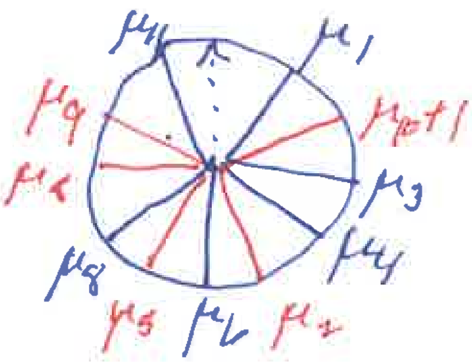
Example: $n=11$ and $j=7$

$c=\{2,5,7,9,10\}$ $\quad$ $c^c=\{1,3,4,6,8,11\}$

$\mu =$



$\mathrm{rot}_c(\mu) =$



$$t^\vee_{c,7} = t^\vee_6 t^\vee_4 t^\vee_3 t^\vee_1 t^\vee_\pi t^\vee_{11-1} t^\vee_{8-1}$$

$$F_{c,7}(y) = y_2 y_7^{-1} - y_2 y_5^{-1}$$

$$F_\mu(c;7) = q^{-\mu_2+\mu_7} t^{v_\mu(2)-v_\mu(7)} - q^{-\mu_2+\mu_5} t^{v_\mu(2)-v_\mu(5)}$$

$$\mathrm{wt}_\mu(c) = t^{-\#\{i \mid \mu_i > \mu_{i+1}\}} \Big(\prod_{\substack{k\in c \\ k\neq n_1}} \frac{1-t}{1-q^m t^m} \Big) \frac{(1-t)}{(1-q q^m t^m)} \cdot \prod_{k\notin c} \mathrm{wt}_\mu(c,k)$$

where

$$\mathrm{wt}_\mu(c,k) = \begin{cases} 0, & \text{if } b(k)=\mu_k, \\ 1, & \text{if } b(k)>\mu_k, \\ t\,\dfrac{(1-q^m t^{m+1})(1-q^m t^{m-1})}{(1-q^m t^m)^2}, & \text{if } b(k)<\mu_k. \end{cases}$$