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CART Seminar, KIAS

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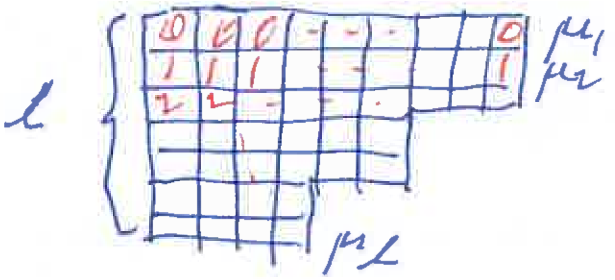
A new definition of integral form Macdonald polynomials

Partitions of n: Let $n \in \mathbb{Z}_{>0}$.

$$\mathcal{P}_n = \left\{ \mu = (\mu_1, \ldots, \mu_\ell) \;\middle|\; \begin{array}{l} \mu_1 \geq \cdots \geq \mu_\ell > 0 \\ \mu_1 + \cdots + \mu_\ell = n \end{array} \right\}$$

and

$$\ell(\mu) = \ell \quad \text{and} \quad n(\mu) = 0\mu_1 + 1\cdot\mu_2 + \cdots + (\ell-1)\mu_\ell$$



$n(\mu)$

Context:

$\tilde{H}_\mu[X;q,t]$ modified Macdonald polynomial

$J_\mu(x;q,t)$ integral form Macdonald polynomial

$$\tilde{H}_\mu[X;q,t] = t^{n(\mu)} J_\mu\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right]$$

These are type GL_n creatures.

Question: Is there a generalization to all Lie types $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$?

Contraction:

$C = (C_{w\pi})$ with $w \in S_n$ and $\pi \in \mathcal{P}_n$.

$$C_{w\pi} = \begin{cases} 0, & \text{if } w \notin S_\pi, \\ 1, & \text{if } w \in S_\pi, \end{cases} \quad \text{where}$$

$S_\pi = S_{\pi_1} \times \cdots \times S_{\pi_\ell} \subseteq S_n$ if $\pi = (\pi_1, \ldots, \pi_\ell)$.

If $n = 3$ then

$$C = \begin{array}{c} \begin{array}{ccc} (1^3) & (21) & (3) \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Then $\mathbb{1}_\pi = \sum_{w \in S_n} C_{w\pi} T_w$ is a "parabolic projector"

in the Hecke algebra $H(S_n)$.

Definition $\tilde{H}_\mu(z,t) = \sum_\pi \mathrm{Card}(Y^{-1}_{I_\pi}(u_\mu))\, \mathbb{1}_\pi$

where

$Y^{-1}_{I_\pi}(u_\mu) = \{ y I_\pi \in G/I_\pi \mid y^{-1} u_\mu y \in I_\pi \}$

is a π-parabolic affine Springer fiber over u_μ.

Expansion

$K(q) = (K_{\nu w}(q))$ with $w \in S_n$ and $\nu \in \mathcal{P}_n$.

If $n=3$ then

$$K(q) = \begin{array}{c|c} & \begin{array}{cccccc} \equiv & \underline{\text{5}} & \overline{\text{x}} & * & \times & \text{x} \end{array} \\ \begin{array}{c} (1^3) \\ (21) \\ (3) \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & q & q-1 & q-1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Properties of $K(q)$

Let
$$\mathcal{W}_\nu = \{ w \in S_n \mid w \text{ has cycle type } \nu \}$$

Then
$$K_{\nu w}(1) = \begin{cases} 0, & \text{if } w \notin \mathcal{W}_\nu, \\ 1, & \text{if } w \in \mathcal{W}_\nu, \end{cases}$$

and the Geck-Rouquier basis of the center of the Hecke algebra is

$$\{ K_\nu \mid \nu \in \mathcal{P}_n \} \quad \text{a basis of } Z(H(S_n))$$

where
$$K_\nu = \sum_{w \in S_n} K_{\nu w}(q)\, q^{-\ell(w)} T_w$$

Definition
$$J_\mu(q,t) = \sum_\nu \mathrm{Card}\left(\mathcal{Y}^{-1}_{I \gamma_\nu I}(u_\mu)\right) K_\nu$$

where
$$\mathcal{Y}^{-1}_{I\gamma_\nu I}(u_\mu) = \{ gI \in G/I \mid g^{-1} u_\mu g \in I \gamma_\nu I \}$$

is an affine Lusztig variety for γ_ν and u_μ

Monomial expansions

$$J_\mu(x;q,t) = \sum_{\nu\in P_n} a_{\mu\nu}(q,t)(1-t)^{\ell(\nu)} m_\nu,$$

$$\tilde{H}_\mu[X;q,t] = \sum_{\pi\in P_n} b_{\mu\pi}(q,t)\, m_\pi,$$

where m_π are monomial symmetric functions.

Let $R_{\nu\pi}(q)$ be given by

$$m_\nu\left[\frac{X}{1-q^{-1}}\right] = \sum_\pi R_{\nu\pi}(q)\, m_\pi.$$

Since

$$\tilde{H}_\mu[X;q,t] = t^{n(\mu)} J_\mu\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right]$$

then

$$b_{\mu\pi}(t,q) = \sum_{\nu\in P_n} q^{n(\mu)} a_{\mu\nu}(t,q^{-1})\, R_{\nu\pi}(q)$$

$$= \sum_{\substack{\nu\in P_n \\ w\in S_n}} q^{n(\mu)} a_{\mu\nu}(t,q^{-1})\, q^{n-\ell(\nu)} K_{\nu w}(q)\, C_{w\pi} \frac{1}{[\pi]!}$$

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Expansion-Contraction = Plethystic transformation

Let
$$[r]! = \frac{(q^r-1)\cdots(q^2-1)(q-1)}{(q-1)^r} \quad \text{and}$$

$$[\pi]! = [\pi_1]!\cdots[\pi_\ell]! \quad \text{if } \pi = (\pi_1, \ldots, \pi_\ell).$$

Theorem

$$R_{\nu\pi}(q) = \sum_{w\in S_n} q^{n-\ell(w)} K_{\nu w}(q)\, C_{w\pi} \frac{1}{[\pi]!}$$

Theorem (Mellit) Let $u_\mu \in GL_n(\mathbb{F}_q)$ be unipotent of Jordan form μ.
Then
$$b_{\mu\pi}(t,q) = \mathrm{Card}(Y_{I_\pi}^{-1}(u_\mu))$$
\# of points of a π-parabolic affine Springer fiber over u_μ

Theorem Let γ_ν be minimal length in W_ν
Then
$$q^{n(\mu)} a_{\mu\nu}(t, q^{-1})\, q^{n-\ell(w)} = \mathrm{Card}(Y_{I\gamma_\nu I}^{-1}(u_\mu))$$
\# of points of an affine Lusztig variety for γ_ν and u_μ

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Loop groups

$$
\begin{array}{ccc}
G = GL_n(\mathbb{F}_q((\epsilon))) & & \\
\cup\!| & & \\
K = GL_n(\mathbb{F}_q[[\epsilon]]) & \xrightarrow{\epsilon = 0} & GL_n(\mathbb{F}_q) \\
\cup\!| & & \cup\!| \\
I_\pi & \longrightarrow & P_\pi = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \\
\cup\!| & & \text{block sizes given by } \pi = (\pi_1, \ldots, \pi_\ell) \\
 & & \cup\!| \\
I & \longrightarrow & B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}
\end{array}
$$

Then the π-parabolic affine Springer fiber is

$$Y^{-1}_{I_\pi}(u_\mu) = \{ gI_\pi \in G/I_\pi \mid g^{-1} u_\mu g \in I_\pi \}$$

and the Lusztig variety for γ_v and u_μ is

$$Y^{-1}_{I\gamma_v I}(u_\mu) = \{ gI \in G/I \mid g^{-1} u_\mu g \in I \gamma_v I \}$$