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Berkeley Seminar
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Loop Groups

$$
\begin{tikzcd}
G = GL_n(\mathbb{F}_q((\epsilon))) \\
K = GL_n(\mathbb{F}_q[[\epsilon]]) \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "\epsilon = 0"] & GL_n(\mathbb{F}_q) \\
I_\pi \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] & P_\pi = \left\{ \begin{pmatrix} \begin{array}{c|c} \ & * \\ 0 & \ \end{array} \end{pmatrix} \right\} \arrow[u, phantom, sloped, "\subseteq"] \\
I \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] & B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}
$$

block sizes given by $\pi = (\pi_1, \ldots, \pi_\ell)$

upper triangular.

$$y^{-1}_{I_\pi}(u_\mu) = \{ yI_\pi \in G/I_\pi \mid y^{-1}u_\mu y \in I_\pi \}$$

$$y^{-1}_{I\gamma_\nu I}(u_\mu) = \{ yI \in G/I \mid y^{-1}u_\mu y \in I\gamma_\nu I \}$$

ETA Modified Macdonald polynomials are not Macdonald polynomials

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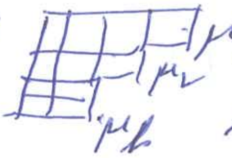
Partitions of n: $\mathcal{P}_n = \{\mu = (\mu_1, \dots, \mu_\ell) \mid \mu_1 \geq \dots \geq \mu_\ell,\ \mu_1 + \dots + \mu_\ell = n\}$

$\ell\Big($  $\Big) = \ell$ and $n(5542) = 0+0+0+0+0$
$+1+1+1+1+1$
$+2+2+2+2$
$+3+3$.

Context: $\tilde{H}_\mu[X; q, t]$ Modified Macdonald polynomial

$J_\mu(x; q, t)$ Macdonald polynomial integral form

$$\tilde{H}_\mu[X; q, t] = t^{n(\mu)} J_\mu\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right]$$

These are type GL_n-creatures.

Question: Is there a generalization to all Lie types: $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$?

Contraction

$C = (c_{w\pi})$ with $w \in S_n$ and $\pi \in P_n$

$$c_{w\pi} = \begin{cases} 0, & \text{if } w \notin S_\pi, \\ 1, & \text{if } w \in S_\pi, \end{cases} \quad \text{where}$$

$S_\pi = S_{\pi_1} \times \cdots \times S_{\pi_\ell} \subseteq S_n$ if $\pi = (\pi_1, \ldots, \pi_\ell)$.

If $n = 3$ then

$$C = \begin{array}{ccc} (1^3) & (21) & (3) \end{array}$$
$$C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$S_{(1^3)} = S_1 \times S_1 \times S_1 \subseteq S_3$

$S_{(21)} = S_2 \times S_1 \subseteq S_3$

$S_{(3)} = S_3$

Then

$$\mathbb{1}_\pi = \sum_{w \in S_n} c_{w\pi} T_w \quad \text{is a "parabolic projector"}$$

in the Hecke algebra $H(S_n)$

Definition
$$\tilde{H}_\mu(q,t) = \sum_{\pi \in P_n} \mathrm{Card}\left(Y^{-1}_{I_\pi}(u_\mu)\right) \mathbb{1}_\pi$$

where $Y^{-1}_{I_\pi}(u_\mu)$ is a π-parabolic affine Springer fiber over u_μ.

<u>Expansion</u>

$$K(q) = (K_{\nu w}(q)) \text{ with } \nu \in \mathcal{P}_n \text{ and } w \in S_n.$$

If $n=3$ then

$$K(q) = \begin{array}{c} \\ (1^3) \\ (21) \\ (3) \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & q & q-1 & q-1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

<u>Properties of $K(q)$</u> Let

$$W_\nu = \{ w \in S_n \mid w \text{ is cycle type } \nu \}$$

Then

$$K_{\nu w}(1) = \begin{cases} 0, & \text{if } w \notin W_\nu, \\ 1, & \text{if } w \in W_\nu, \end{cases}$$

and with

$$K_\nu = \sum_{w \in S_n} K_{\nu w}(q)\, q^{-\ell(w)} T_w$$

then

$\{K_\nu \mid \nu \in \mathcal{P}_n\}$ is the <u>Geck-Rouquier basis</u> of $Z(H(S_n))$

<u>Definition:</u> $$J_\mu(q,t) = \sum_{\nu \in \mathcal{P}_n} \mathrm{Card}\left(Y^{-1}_{I\gamma_\nu I}(u_\mu)\right) K_\nu$$

where

$Y^{-1}_{I\gamma_\nu I}(u_\mu)$ is an affine Lusztig variety for γ_ν and u_μ.

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Monomial expansions

$$J_\mu(x;q,t) = \sum_{\nu\in P_n} a_{\mu\nu}(q,t)(1-t)^{\ell(\nu)} m_\nu$$

$$\tilde{H}_\mu[X;q,t] = \sum_{\pi\in P_n} d_{\mu\pi}(q,t)\, m_\pi$$

where m_π are monomial symmetric functions.

Let $R_{\nu\pi}(q)$ be given by

$$m_\nu\left[\frac{X}{1-q^{-1}}\right] = \sum_\pi R_{\nu\pi}(q)\, m_\pi$$

Since

$$\tilde{H}_\mu[X;q,t] = t^{n(\mu)} J_\mu\left(\cancel{\tfrac{X}{1-t^{-1}}};q,t^{-1}\right)$$

then

$$d_{\mu\pi}(t,q) = \sum_{\nu\in P_n} q^{n(\mu)} a_{\mu\nu}(t,q^{-1})\, R_{\nu\pi}(q)$$

$$= \sum_{\substack{\nu\in P_n\\ w\in S_n}} q^{n(\mu)} a_{\mu\nu}(t,q^{-1})\, q^{n-\ell(\nu)} K_{\nu w}(q)\, C_{w\pi} \frac{1}{[\pi]!}$$

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Expansion-Contraction

Let $[r]! = \dfrac{(q^r-1)\cdots(q^2-1)(q-1)}{(q-1)^r}$ and

$[\pi]! = [\pi_1]!\cdots[\pi_\ell]!$ if $\pi = (\pi_1,\ldots,\pi_\ell)$.

Theorem

$$R_{\nu\pi}(q) = \sum_{w\in S_n} q^{n-\ell(\nu)} K_{\nu w}(q) C_{w\pi} \frac{1}{[\pi]!}$$

Theorem (Mellit) Let $u_\mu \in GL_n(\mathbb{F}_q)$ be unipotent of Jordan form μ. Then

$$b_{\mu\pi}(t,q) = \mathrm{Card}(\mathcal{Y}^{-1}_{I_\pi}(u_\mu)).$$ # of points in an affine Springer fiber

Theorem Let γ_ν be minimal length in W_ν. Then

$$q^{n(\mu)} a_{\mu\nu}(t,q^{-1}) q^{n-\ell(\nu)} = \mathrm{Card}(\mathcal{Y}^{-1}_{I\gamma_\nu I}(u_\mu))$$

# of points in an affine Lusztig variety.