03.09.2024 Number theory Seminar UniMelb A. Ram

(1)

Step one towards $\Gamma\backslash G/K$

$$\Gamma\backslash G/K = \{\Gamma g K \mid g\in G\}$$

where $\Gamma g K = \{\gamma g k \mid \gamma\in\Gamma, k\in K\}$ (double coset)

We need a representative for each double coset.

Familiar cases

(a) $GL_n = \bigsqcup_{w\in S_n} BwB$ row reduction and pivots.

(b) $M_{t\times s}(\mathbb{Z}) = \bigsqcup_{d\in D} GL_t(\mathbb{Z})\, d\, GL_s(\mathbb{Z})$ Smith Normal form

(c) $GL_n(\mathbb{R}) = \bigsqcup_{a\in A} O_n(\mathbb{R})\, a\, O_n(\mathbb{R})$ Singular value decomposition.

All of these are done with row reduction.

Remark: If $G = SL_2(\mathbb{R})$ and $K = SO_2(\mathbb{R})$ then G/K is the upper half plane.

In our examples we want

$G = GL_n(\mathbb{Q}_p)$, $K = GL_n(\mathbb{Z}_p)$ or $G = GL_n(\mathbb{F}_p((t)))$, $K = GL_n(\mathbb{F}_p[[t]])$ or $G = GL_n(\mathbb{R})$, $K = O_n(\mathbb{R})$.

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Number systems

$$\mathbb{F}_p = \{0, 1, \ldots, p-1\}.$$

$$\mathbb{F}_p((t)) = \{a_{-\ell} t^{-\ell} + a_{-\ell+1} t^{-\ell+1} + \cdots \mid a_i \in \mathbb{F}_p, \ell \in \mathbb{Z}\}$$

$\cup|$

$$\mathbb{F}_p[[t]] = \{a_0 + a_1 t + a_2 t^2 + \cdots \mid a_i \in \mathbb{F}_p\}$$

$\cup|$

$$\mathbb{F}_p[t] = \left\{a_0 + a_1 t + a_2 t^2 + \cdots \;\middle|\; \begin{array}{l} a_i \in \mathbb{F}_p \text{ and all but a} \\ \text{finite number of } a_i \text{ are } 0 \end{array}\right\}$$

$$\mathbb{Q}_p = \{a_{-\ell} p^{-\ell} + a_{-\ell+1} p^{-\ell+1} + \cdots \mid a_i \in \mathbb{F}_p, \ell \in \mathbb{Z}\}$$

$\cup|$

$$\mathbb{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \cdots \mid a_i \in \mathbb{F}_p\}$$

$\cup|$

$$\mathbb{Z} = \left\{a_0 + a_1 p + a_2 p^2 + \cdots \;\middle|\; \begin{array}{l} a_i \in \mathbb{F}_p \text{ and all but a finite} \\ \text{number of } a_i \text{ are } 0 \end{array}\right\}$$

$$\mathbb{R}_{30} = \{a_{-\ell} (\tfrac{1}{10})^{-\ell} + a_{-\ell+1} (\tfrac{1}{10})^{-\ell+1} + \cdots \mid a_i \in \mathbb{F}_{10}, \ell \in \mathbb{Z}\}$$

$\cup|$

$$\mathbb{R}_{[0,10)} = \{a_0 + a_1 (\tfrac{1}{10}) + a_2 (\tfrac{1}{10})^2 + \cdots \mid a_i \in \mathbb{F}_{10}\}$$

$\cup|$

$$\mathbb{Q}^{sm}_{[0,10)} = \left\{a_0 + a_1 (\tfrac{1}{10}) + a_2 (\tfrac{1}{10})^2 + \cdots \;\middle|\; \begin{array}{l} a_i \in \mathbb{F}_p \text{ and all but} \\ \text{a finite number of} \\ a_i \text{ are } 0 \end{array}\right\}$$

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Geometric Representation Theory

$\mathbb{F}_p = \mathbb{C}$.

$G = GL_n(\mathbb{C}((t)))$
$\cup|$
$K = GL_n(\mathbb{C}[[t]])$
$\cup|$
$I = \{g(t) \in K \mid g(0) \text{ is upper triangular}\}$

and their "Lie algebras"

$Lie(G) = M_n(\mathbb{C}((t)))$
$\cup|$
$Lie(K) = M_n(\mathbb{C}[[t]])$
$\cup|$
$Lie(I) = \{a(t) \in Lie(K) \mid a(0) \text{ is upper triangular}\}$.

Then

G is the <u>loop group</u>

G/K is the <u>loop Grassmannian</u>

$I\backslash G$ is the <u>affine flag variety</u>

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<u>n-periodic matrices</u>

Fix $n \in \mathbb{Z}_{>0}$. A <u>n-periodic matrix</u> is

$$a \in M_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{F}_p) \quad \text{such that}$$

(a) (n-periodicity)

if $i, j \in \mathbb{Z}$ then $a_{i+n, j+n} = a_{ij}$

(b) (left finite in each row).

if $r \in \mathbb{Z}$ then there exists $m \in \mathbb{Z}$ such that if $c < m$ then $a_{rc} = 0$.

Let

$$M_{nper}(\mathbb{F}_p) = \{n \text{ periodic matrices with entries in } \mathbb{F}_p\}$$

Then

$$M_{nper}(\mathbb{F}_p) \xrightarrow{\sim} M_n(\mathbb{F}_p((t)))$$

$$a \longmapsto a(t)$$

where

$$a(t)_{ij} = \sum_{\ell \in \mathbb{Z}} a_{i, j+\ell n} t^{\ell} \quad \text{for } i, j \in \{1, \dots, n\}.$$

Then $\mathrm{Lie}(G) = M_{nper}(\mathbb{F}_p)$ and

$$\mathrm{Lie}(I) = \{\text{upper triangular } n\text{-periodic matrices with entries in } \mathbb{F}_p\}$$

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Example

$$a(t) = \begin{pmatrix} 2+3t^2 & t^{-2}+4 \\ t^{-2}+t^{-1} & 1+t+t^2+\cdots \end{pmatrix} \in M_2(\mathbb{F}_7((t))).$$

and the corresponding $a \in M_{2\text{per}}(\mathbb{F}_7)$ is

$$\begin{array}{l}
\cdots\ 00\ 01\ 00\ 24\ 00\ 30\ 00\ 00 \\
\quad 00\ {-10}\ {-10}\ 01\ 01\ 01\ 01\ 01\ \cdots \\
\quad\quad \cdots\ 00\ 01\ 00\ 24\ 00\ 30\ 00\ 00 \\
\quad\quad\quad 00\ {-10}\ {-10}\ 01\ 01\ 01\ 01\ 01\ \cdots \\
\quad\quad\quad\quad \cdots\ 00\ 01\ 00\ 24\ 00\ 30\ 00\ 00 \\
\quad\quad\quad\quad\quad 00\ {-10}\ {-10}\ 01\ 01\ 01\ 01\ 01\ \cdots
\end{array}$$

An <u>n-periodic permutation</u> is an n-periodic matrix such that

(a) each row and each column contain exactly one nonzero entry

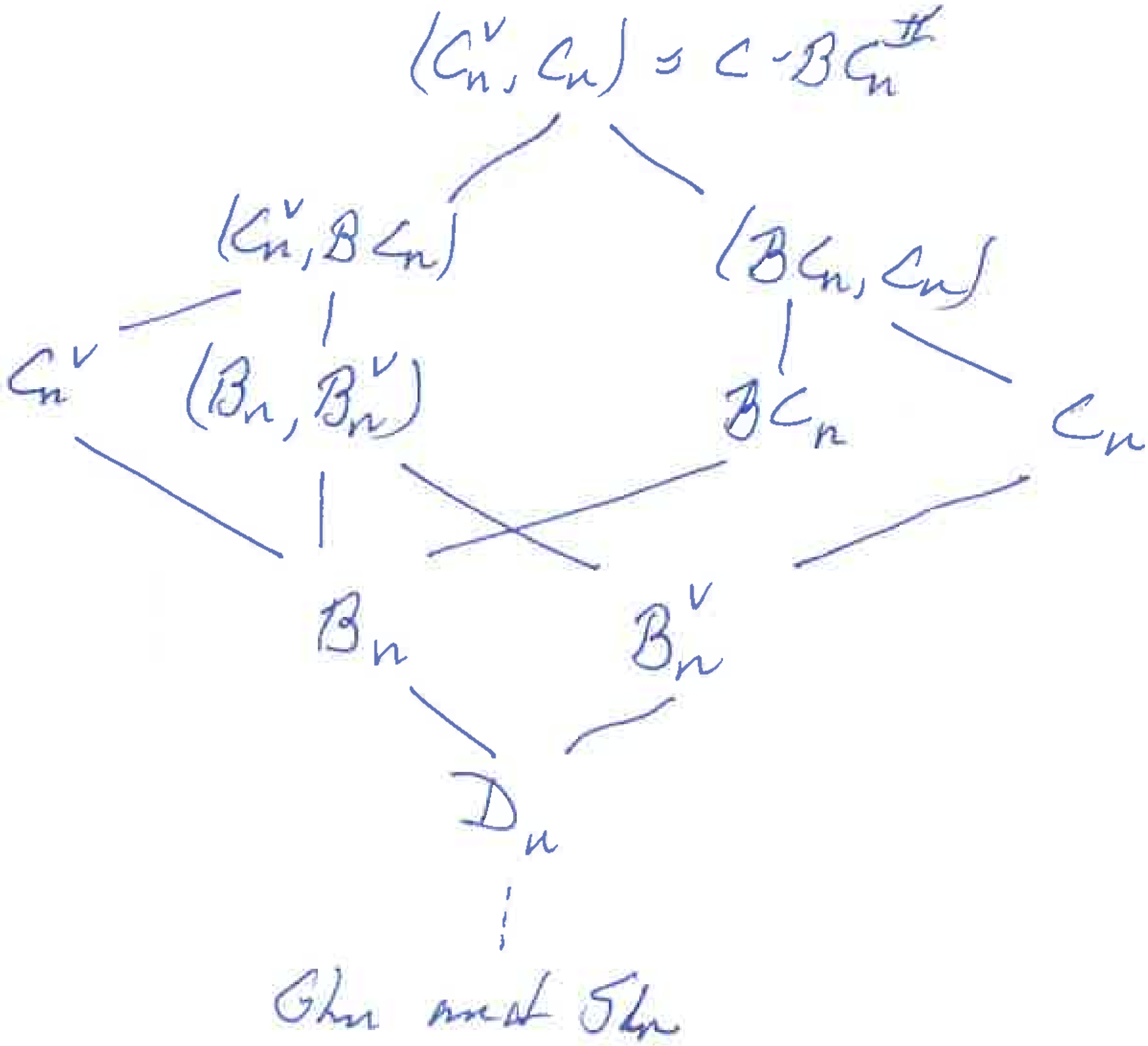
(b) the nonzero entries are 1.

The <u>affine Weyl group</u> (of type GL_n) is the set of n-periodic matrices.

Groups of classical type

Orthogonal, symplectic, unitary, GL_n and SL_n.

THE LIST of Bruhat-Tits



<u>GOAL</u> Describe the G of classical type by n-periodic matrices so that row reduction is available as a tool for double coset analysis.