07.01.2025
ART Seminar

Vector Calculus in 4 pages

Page 1: Graphing

Volumes, surfaces, curves.

Page 2: Derivatives

Tangent spaces and the Chain rule

Page 3: Integrals Volumes/Surfaces/lines
Work and flux.

Page 4: The fundamental Theorem of Calculus

$$\int_{\partial S} \omega = \int_S d\omega$$

References

(1) earth.nullschool.net

(2) H. Rogers / S. Helgason

(3) Abraham / Marsden / Ratiu

(4) 2017 and 2018 Melbourne Univ.

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Page 1: Graphing

Curves: $c: \mathbb{R} \longrightarrow \mathbb{R}^3$

$t \longmapsto (c_1(t), c_2(t), c_3(t))$

$c(t) = p + tv$

Line: $p, v \in \mathbb{R}^3$

$c(t) = (t, \sqrt{a^2 - t^2}, 0)$

Semicircle radius a

$c(t) = (t, t^2, 0)$

Parabola

$c(t) = (2\cos(t), 2\sin(t), 3t)$

helix

$c(t) = (a(t - \sin t), b(1 - \cos t), 0)$ cardiod

Surfaces $\Phi: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$(u,v) \longmapsto (\varphi_1(u,v), \varphi_2(u,v), \varphi_3(u,v))$

hemisphere of radius a $\Phi(u,v) = (u, v, \sqrt{a^2 - u^2 - v^2})$

$x^2 + y^2 + z^2 = a^2$

Sphere of radius a

$x^2 + y^2 = z^2$

cone

$x^2 + y^2 = z$

paraboloid

$x^2 - y^2 = z$

hyperboloid

$x^2 + y^2 = a^2$

cylinder of radius a

Page 1: The chain rule

$g: \mathbb{R}^n \to \mathbb{R}^m$ gives $D_g(p): T_p \to T_{g(p)}$
$(x_1, \dots, x_n) \mapsto (g_1, \dots, g_m)$

given by

$$D_g(p) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(p) & \cdots & \frac{\partial g_1}{\partial x_n}(p) \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1}(p) & \cdots & \frac{\partial g_m}{\partial x_n}(p) \end{pmatrix}$$

with respect to the basis $\{\partial_{x_1}, \dots, \partial_{x_n}\}$ of T_p and the basis $\{\partial_{y_1}, \dots, \partial_{y_m}\}$ of $T_{g(p)}$

<u>Chain rule:</u> If $\varphi: \mathbb{R}^s \to \mathbb{R}^n$ and $g: \mathbb{R}^n \to \mathbb{R}^m$
$(u_1, \dots, u_s) \mapsto (x_1, \dots, x_n)$ $\quad$ $(x_1, \dots, x_n) \mapsto (g_1, \dots, g_m)$

then

$$D_{g\circ\varphi}(p) = D_g(\varphi(p))\, D_\varphi(p)$$

is a generalization of $\frac{dg}{du} = \frac{dg}{dx}\frac{dx}{du}$

Volume integrals and the Jacobian

If $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ and $g: \mathbb{R}^n \to \mathbb{R}^m$ then
$(u_1, \dots, u_n) \mapsto (x_1, \dots, x_n)$ $\quad$ $(x_1, \dots, x_n) \mapsto (g_1, \dots, g_m)$

$$\int g(x_1, \dots, x_n)\, dx_1 \cdots dx_n = \int g(u_1, \dots, u_n)\, |\mathrm{Jac}(\varphi)|\, du_1 \cdots du_n$$

where $|\mathrm{Jac}(\varphi)| = |\det(D_\varphi)|$.

This is a generalization of

$$\int g(x)\, dx = \int g(u) \frac{dx}{du}\, du$$

Page 3: Integration

Curves: $c: \mathbb{R} \longrightarrow \mathbb{R}^3$
$t \mapsto (c_1(t), c_2(t), c_3(t))$ has tangent vector

$\partial_c = \frac{\partial c_1}{\partial t} \partial_{x_1} + \frac{\partial c_2}{\partial t} \partial_{x_2} + \frac{\partial c_3}{\partial t} \partial_{x_3}$ and $ds = |\partial_c| dt$

gives

$$\int_c f(c_1(t), c_2(t), c_3(t))\, ds \quad (\text{line integral})$$

Surfaces: $\Phi: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
$(u,v) \longmapsto (\varphi_1(u,v), \varphi_2(u,v), \varphi_3(u,v))$ has

tangent vectors

$\partial_u = \frac{\partial \varphi_1}{\partial u} \partial_{x_1} + \frac{\partial \varphi_2}{\partial u} \partial_{x_2} + \frac{\partial \varphi_3}{\partial u} \partial_{x_3}$
and $dS = |\partial_u \times \partial_v| du\, dv$
$\partial_v = \frac{\partial \varphi_1}{\partial v} \partial_{x_1} + \frac{\partial \varphi_2}{\partial v} \partial_{x_2} + \frac{\partial \varphi_3}{\partial v} \partial_{x_3}$

gives

$$\int_S f(\varphi_1(u,v), \varphi_2(u,v), \varphi_3(u,v))\, dS \quad (\text{surface integral})$$

Let $F = F_1 \partial_{x_1} + F_2 \partial_{x_2} + F_3 \partial_{x_3}$ in $\mathbb{R}^3$. Then

(Work integral) $\int_c \langle F | \hat{T} \rangle ds$ with $\hat{T} = \frac{1}{|\partial_c|} \partial_c$

(Flux integral) $\int_S \langle F | \hat{n} \rangle dS$ with $\hat{n} = \frac{1}{|\partial_u \times \partial_v|} \partial_u \times \partial_v$

Page 4: Fund. Theorem of Calculus ART Seminar

$$0 \to \mathcal{O}_{\mathbb{R}^3} \xrightarrow{d} \Omega^1_{\mathbb{R}^3} \xrightarrow{d} \Omega^2_{\mathbb{R}^3} \xrightarrow{d} \Omega^3_{\mathbb{R}^3} \to 0$$

Where
$$\mathcal{O}_{\mathbb{R}^3} = \{ f : \mathbb{R}^3 \to \mathbb{R} \}$$

$$\Omega^1_{\mathbb{R}^3} = \{ F_1\, dx_1 + F_2\, dx_2 + F_3\, dx_3 \mid F_1, F_2, F_3 \in \mathcal{O}_{\mathbb{R}^3} \}$$

$$\Omega^2_{\mathbb{R}^3} = \{ G_1\, dx_1 \wedge dx_2 + G_2\, dx_1 \wedge dx_3 + G_3\, dx_2 \wedge dx_3 \mid G_1, G_2, G_3 \in \mathcal{O}_{\mathbb{R}^3} \}$$

$$\Omega^3_{\mathbb{R}^3} = \{ h\, dx_1 \wedge dx_2 \wedge dx_3 \mid h \in \mathcal{O}_{\mathbb{R}^3} \}$$

with
$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3$$

$$dF = \left(\frac{\partial F_2}{\partial x_2} \frac{\partial F_3}{\partial x_3} - \frac{\partial F_3}{\partial x_2} \frac{\partial F_2}{\partial x_3} \right) dx_2 \wedge dx_3$$
$$+ \left(\frac{\partial F_1}{\partial x_1} \frac{\partial F_3}{\partial x_3} - \frac{\partial F_3}{\partial x_3} \frac{\partial F_1}{\partial x_1} \right) dx_1 \wedge dx_3$$
$$+ \left(\frac{\partial F_1}{\partial x_1} \frac{\partial F_2}{\partial x_2} - \frac{\partial F_2}{\partial x_1} \frac{\partial F_1}{\partial x_2} \right) dx_1 \wedge dx_2$$

$$dG = \left(\frac{\partial G}{\partial x_1} + \frac{\partial G}{\partial x_2} + \frac{\partial G}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3$$

For $f \in \mathcal{O}_{\mathbb{R}^3}$, $F \in \Omega^1_{\mathbb{R}^3}$, $G \in \Omega^2_{\mathbb{R}^3}$,

$$\int_{\partial C} f = \int_C df, \qquad \int_{\partial S} F = \int_S dF, \qquad \int_{\partial V} G = \int_V dG$$

generalizing

$$\int_{\mathbb{R}_{[a,b]}} \frac{df}{dx}\, dx = f(b) - f(a).$$