14.01.2015 (1)
ART Seminar
A. Ram

Formulas for Macdonald Polynomials

Electronic Macdonald Polynomials

For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$ let $v_\mu \in S_n$ be given by

$$v_\mu(i) = 1 + \#\{i' \in \{1, \ldots, i-1\} \mid \mu_{i'} \leq \mu_i\} + \#\{i' \in \{i+1, \ldots, n\} \mid \mu_{i'} < \mu_i\}.$$

For $i \in \{1, \ldots, n-1\}$ let

$$\partial_i = (1 + s_i) \frac{1}{x_i - x_{i+1}}.$$

Then $E_\mu \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is determined by

(E0) $E_{(0, \ldots, 0)} = 1$,

(E1) $E_{(\mu_n+1, \mu_1, \ldots, \mu_{n-1})} = q^{\mu_n} x_1 E_\mu(x_2, \ldots, x_n, q^{-1} x_1)$

(E2) If $\mu_i > \mu_{i+1}$ then

$$E_{s_i \mu} = \left(\partial_i x_i - t x_i \partial_i + \frac{(1-t)\, q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}{1 - q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}} \right) E_\mu$$

where

$$s_i(\mu_1, \ldots, \mu_n) = (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2}, \ldots, \mu_n).$$

14.01.2015 (2)
ART Seminar
A. Ram

The alcove walk formula

For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$ let

$$s_i(\mu_1, \ldots, \mu_n) = (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2}, \ldots, \mu_n)$$

$$s_\pi(\mu_1, \ldots, \mu_n) = (\mu_n + 1, \mu_1, \ldots, \mu_{n-1})$$

Let $s_{i_1} \cdots s_{i_\ell}$ (with $i_1, \ldots, i_\ell \in \{1, \ldots, n-1\} \cup \{\pi\}$) be minimal length such that

$$s_{i_1} \cdots s_{i_\ell}(0, \ldots, 0) = (\mu_1, \ldots, \mu_n)$$

An alcove walk is a subset

$$p \subseteq \{k \in \{1, \ldots, \ell\} \mid i_k \neq \pi\}.$$

Let

$$AW(\mu) = \{\text{alcove walks for } s_{i_1} \cdots s_{i_\ell}\}.$$

For $k \in \{1, \ldots, \ell\}$ let

$$p_k = s_{i_1} \cdots \cancel{s_{i_{p_1}}} \cdots \cancel{s_{i_{p_r}}} \cdots s_{i_k}$$

(p_k is the subword of $s_{i_1} \cdots s_{i_k}$ with the factors corresponding to p removed)

where $\{p_1, \ldots, p_r\} = p \cap \{1, \ldots, k\}$.

Let

$$\bar{p}_k = p_k \Big|_{s_\pi = s_1 \cdots s_{n-1}}.$$

Let $(\beta_1^\vee, \ldots, \beta_\ell^\vee)$ be the sequence given by

$$\beta_k^\vee = s_{i_\ell}^{-1} \cdots s_{i_{k+1}}^{-1}(\varepsilon_{i_k}^\vee - \varepsilon_{i_{k+1}}^\vee), \quad \text{where}$$

$$s_\pi^{-1}\varepsilon_1^\vee = \varepsilon_n^\vee + K, \qquad s_\pi^{-1}\varepsilon_i^\vee = \varepsilon_{i-1}^\vee \quad \text{and} \quad s_\pi^{-1}K = K$$

$$s_i^{-1}\varepsilon_i^\vee = \varepsilon_{i+1}^\vee,$$

$$s_i^{-1}\varepsilon_{i+1}^\vee = \varepsilon_i^\vee, \qquad s_i^{-1}\varepsilon_j^\vee = \varepsilon_j^\vee \quad \text{and} \quad s_i^{-1}K = K.$$

Let

$$sh(\varepsilon_i^\vee - \varepsilon_j^\vee - rK) = r \quad \text{and} \quad ht(\varepsilon_i^\vee - \varepsilon_j^\vee - rK) = j - i.$$

For $k \in \{1, \ldots, \ell\}$ define

$$wt_p(k) = \begin{cases} \dfrac{1-t}{1 - q^{sh(-\beta_k^\vee)} t^{ht(-\beta_k^\vee)}}, & \text{if } k \in p \text{ and } p_{k-1} s_{i_k} < p_{k-1}, \\[2ex] \dfrac{(1-t)\, q^{sh(-\beta_k^\vee)} t^{ht(-\beta_k^\vee)}}{1 - q^{sh(-\beta_k^\vee)} t^{ht(-\beta_k^\vee)}}, & \text{if } k \in p \text{ and } p_{k-1} s_{i_k} > p_{k-1}, \\[2ex] 1, & \text{if } i_k \neq \pi \text{ and } k \notin p, \\ x_{p_{k-1}(1)}, & \text{if } i_k = \pi. \end{cases}$$

Define

$$wt(p) = \prod_{k=1}^{\ell} wt_p(k)$$

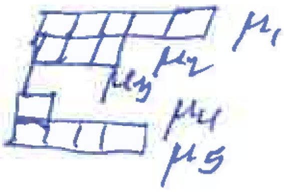
Then

$$E_\mu = \sum_{p \in AW(\mu)} wt(p)$$

The Non Attacking Fillings formula

Identify μ with a set of boxes

$$\left\{ (r,c)\in\mathbb{Z}^2 \;\middle|\; \begin{matrix} r\in\{1,\dots,n\} \\ c\in\{1,\dots,\mu_r\} \end{matrix} \right\} =$$



so that μ has μ_r boxes in row r.

Let $\hat{\mu} = \{(1,0),\dots,(n,0)\}\cup\mu$. Define

$$\mathrm{attack}_\mu(r,c) = \left\{ \begin{matrix} (1,c),\dots,(r-1,c) \\ (r+1,c),\dots,(n,c-1) \end{matrix} \right\} \cap \hat{\mu}$$

A <u>nonattacking filling of μ</u> is $T:\hat{\mu}\to\{1,\dots,n\}$ such that

(a) If $r\in\{1,\dots,n\}$ then $T(r,0)=r$.

(b) If $a\in \mathrm{attack}_\mu(b)$ then $T(a)\neq T(b)$

Let

$$NAF(\mu) = \{\text{nonattacking fillings of } \mu\}.$$

Define

$$\mathrm{Nl}_\mu(b) = \{(r,c+k) \mid k\in\mathbb{Z}_{>0}\}\cap\mu$$

$$\mathrm{Na}_\mu(b) = \{a\in\mathrm{attack}_\mu(b) \mid \#\mathrm{Nl}_\mu(a) \le \#\mathrm{Nl}_\mu(b)\}$$

$$\mathrm{dwn}_T(b) = \{a\in\mathrm{Na}_\mu(b) \mid T(b-n) > T(a) > T(b)\}$$

where

$$b-n = (r,c-1) \quad\text{if}\quad b=(r,c).$$

ART Seminar
A. Ram

Let

$$wt(T) = \prod_{b\in\mu} wt_T(b)$$

where

$$wt_T(b) = \begin{cases} \dfrac{(1-t)\,t^{\#bwn_T(b)}}{1-q^{\#N\ell(b)+1}t^{\#Na(b)+1}}\, x_{T(b)}, & \text{if } T(b-n) > T(b), \\[2ex] \dfrac{(1-t)\,t^{\#bwn_T(b)}\,q^{\#N\ell(b)+1}t^{\#Na(b)+1}}{1-q^{\#N\ell(b)+1}t^{\#Na(b+1)}}\, x_{T(b)}, & \text{if } T(b-n) < T(b), \\[2ex] x_{T(b)}, & \text{if } T(b-n) = T(b). \end{cases}$$

Then

$$E_\mu = \sum_{T\in NAF(\mu)} wt(T).$$

Bosonic Macdonald Polynomials

If $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$ let

$$P_\lambda = \frac{1}{W_\lambda(t)} \sum_{w \in S_n} w\left(E_\lambda \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j}\right)$$

where

$W_\lambda(t)$ is such that the coefficient of $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in P_λ is 1.

A <u>SSYT of shape λ filled from $\{1, \ldots, n\}$</u> is a function $T : \{\text{boxes of } \lambda\} \to \{1, \ldots, n\}$ such that

(a) If $(r,c), (r, c+1) \in \lambda$ then $T(r,c) \leq T(r, c+1)$

(b) If $(r,c), (r+1, c) \in \lambda$ then $T(r,c) < T(r+1, c)$.

Let

$$B(\lambda) = \left\{ \begin{array}{l} \text{SSYT of shape } \lambda \\ \text{filled from } \{1, \ldots, n\} \end{array} \right\}$$

1	1	1	1	2	3	3
2	2	3	4			
3	3	4				

Define

$$x^T = x_1^{\#1\text{s in } T} x_2^{\#2\text{s in } T} \cdots x_n^{\#n\text{s in } T}$$

Define for $b=(r,c)$ in λ, $i\in\{1,\dots,n\}$, ART Seminar

A. Ram

$$\mathrm{arm}_\lambda(b)=\{(r,c')\in\lambda \mid c'>c\}$$

$$\mathrm{leg}_\lambda(b)=\{(r',c)\in\lambda \mid r'>r\}$$

$$a_T(b,<i)=\{b'\in\mathrm{arm}_\lambda(b) \mid T(b')<i\}$$

$$l_T(b,<i)=\{b'\in\mathrm{leg}_\lambda(b) \mid T(b')<i\}$$

$$h_T(b,<i)=\frac{1-tq^{\#a_T(b,<i)}t^{\#l_T(b,<i)}}{1-q\,q^{\#a_T(b,<i)}t^{\#l_T(b,<i)}}$$

and

$$\psi_T(b)=\prod_{\substack{i\in T(\mathrm{arm}_\lambda(b))\\ i\notin T(\mathrm{leg}_\lambda(b))}}\frac{h_T(b,<i)}{h_T(b,\le i)}$$

Let

$$\psi_T(q,t)=\prod_{b\in\lambda}\psi_T(b).$$

Then

$$P_\lambda=\sum_{T\in B(\lambda)}\psi_T(b)\,x^T$$