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Symmetric Functions

S_n acts on $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

$$B = \mathbb{C}[X]^{S_n} = \{ f \in \mathbb{C}[X] \mid \text{if } w \in S_n \text{ then } wf = f \}$$

$$F = \mathbb{C}[X]^{\det} = \{ f \in \mathbb{C}[X] \mid wf = \det(w) f, \text{ for } w \in S_n \}.$$

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ let $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$

$\rho = (n-1, n-2, \ldots, 2, 1, 0)$ and $x^\rho = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$

$$p_0 = \sum_{w \in S_n} w, \qquad m_\lambda = (\text{const})\, p_0 x^\lambda$$

$$e_0 = \sum_{w \in S_n} \det(w) w, \qquad a_\lambda = (\text{const})\, e_0 x^\lambda.$$

As $\mathbb{C}[X]^{S_n}$-modules

$$\mathbb{C}[X]^{S_n} = B \xrightarrow{\sim} F = a_\rho \mathbb{C}[X]^{S_n}$$
$$f \longmapsto a_\rho f$$

naive basis $\quad p_0 x^\lambda \overset{\circ}{=} m_\lambda$

Schur function $s_\lambda \longmapsto a_{\lambda+\rho} \overset{\circ}{=} e_0 x^\lambda \quad$ naive basis

and $\quad a_\rho = \prod_{1 \le i < j \le n} (x_j - x_i)$

Macdonald polynomials $q, t^{\frac{1}{2}} \in \mathbb{C}^\times$

s_i is transposition switching i and $i+1$.

$$\partial_i = \frac{1}{x_i - x_{i+1}}(1 - s_i) \qquad \left(\text{i.e. } \partial_i f = \frac{f - s_i f}{x_i - x_{i+1}}\right)$$

operators on $\mathbb{C}[X]$, for $i \in \{1, \ldots, n-1\}$

The electronic Macdonald polynomials

$E_\mu = E_{(\mu_1, \ldots, \mu_n)}(x_1, \ldots, x_n; q, t)$ for $\mu \in \mathbb{Z}^n$ are given by

(a) $E_{(0, \ldots 0)} = 1$

(b) $E_{(\mu_n + 1, \mu_1, \ldots, \mu_{n-1})} = q^{\mu_n} x_1 E_\mu(x_2, \ldots, x_n, q^{-1} x_1)$

(c) If $\mu_i > \mu_{i+1}$ then

$$E_{s_i \mu} = \left(\partial_i x_i - t x_i \partial_i + \frac{(1-t)\, q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}{1 - q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}\right) E_\mu$$

where

$$v_\mu(j) = \#\{j' < j \mid \mu_{j'} \le \mu_j\} + \#\{j' > j \mid \mu_{j'} < \mu_j\} + 1.$$

Then

$$\{E_\mu(x; q, t) \mid \mu \in \mathbb{Z}^n\} \text{ is a basis of } \mathbb{C}[X].$$

Bosonic and Fermionic

$$T_i = \partial_i x_i - t x_i \partial_i, \quad \text{for } i \in \{1, \ldots, n-1\},$$

operators on $\mathbb{C}[X]$. For $w \in S_n$ and $w = s_{i_1} \cdots s_{i_\ell}$ of minimal length let

$$T_w = T_{i_1} \cdots T_{i_\ell} \quad \text{and} \quad \ell(w) = \ell.$$

Let $\ell(w_0) = \frac{1}{2} n(n-1)$ and

$$\mathbb{1}_0 = \sum_{w \in S_n} (t^{\frac{1}{2}})^{\ell(w) - \ell(w_0)} T_w \qquad \text{Bosonic symmetrizer}$$

$$\varepsilon_0 = \sum_{w \in S_n} (-t^{-\frac{1}{2}})^{\ell(w) - \ell(w_0)} T_w \qquad \text{Fermionic symmetrizer}$$

Let

$$P_\lambda(x; q, t) = (\text{const}) \, \mathbb{1}_0 E_\lambda \qquad \text{Bosonic Macdonald polynomial}$$

$$A_\lambda(x; q, t) = (\text{const}) \, \varepsilon_0 E_\lambda \qquad \text{Fermionic Macdonald polynomial.}$$

Then

$$B_{q,t} = \{ f \in \mathbb{C}[X] \mid T_i f = t^{\frac{1}{2}} f, \text{ for } i \in \{1, \ldots, n-1\} \}$$

$$F_{q,t} = \{ f \in \mathbb{C}[X] \mid T_i f = (-t^{-\frac{1}{2}}) f, \text{ for } i \in \{1, \ldots, n-1\} \}.$$

Boson-Fermion correspondence

$\mathcal{B}_{q,t}$ has basis $\{P_\lambda(x;q,t) \mid \lambda_1 \geq \cdots \geq \lambda_n\}$

$F_{q,t}$ has basis $\{A_{\lambda+\rho}(x;q,t) \mid \lambda_1 \geq \cdots \geq \lambda_n\}$

Then, as $\mathbb{C}[X]^{S_n}$-modules

$$\mathbb{C}[X]^{S_n} = \mathcal{B}_{q,t} \xrightarrow{\sim} F_{q,t} = A_\rho\, \mathbb{C}[X]^{S_n}$$
$$f \longmapsto A_\rho f$$

$\mathbb{1}_0 E_\lambda \doteq P_\lambda(q,t)$

$$P_\lambda(q,q^t) \longmapsto A_{\lambda+\rho}(q,t) \doteq \varepsilon_0 E_{\lambda+\rho}$$

and
$$A_\rho(x;q,t) = \prod_{1\leq i<j\leq n} (x_j - t x_i)$$

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Geometric Satake: $q=0$.

Affine Hecke algebra H: the algebra generated by $x_1, \dots, x_n$ and $T_1, \dots, T_{n-1}$

The finite Hecke algebra H_{fin} is generated by the operators $T_1, \dots, T_{n-1}$.

$$H \text{ has basis } \{x^\mu T_w \mid \mu \in \mathbb{Z}^n, w \in S_n\}$$

$$H_{fin} \text{ has basis } \{T_w \mid w \in S_n\}.$$

Then

$$\mathbb{C}[X] \cong H \mathbb{1}_0 \quad \text{as } H\text{-modules.}$$

and

$$\begin{array}{ccccc} & & K(\mathrm{Perv}_K(G/K)) = \text{spherical Hecke alg} & & K(\mathrm{Whitt}(G/K)) \\ K(\mathrm{Rep}(G^\vee)) & & \| & & \| \\ \| & & \mathbb{1}_0 H \mathbb{1}_0 & & \varepsilon_0 H \mathbb{1}_0 \\ & & \| & & \| \\ \mathbb{C}[X]^{S_n} = Z(H) & \longrightarrow & B_{0,t} & \longrightarrow & F_{0,t} \\ & & f\mathbb{1}_0 & \longmapsto & A_\rho f \mathbb{1}_0 \end{array}$$

Spherical function $P_\lambda(0,t) \longmapsto \mathbb{1}_0 X^\lambda \mathbb{1}_0$

$$s_\lambda = P_\lambda(0,0) \longmapsto C_\lambda \longmapsto A_{\lambda+\rho} = \varepsilon_0 X^{\lambda+\rho} \mathbb{1}_0$$

Kazhdan-Lusztig basis element

where $G = G(\mathbb{C}((t)))$ and $K = G(\mathbb{C}[[t]])$

or $G = G(\mathbb{Q}_p)$ and $K = G(\mathbb{Z}_p)$.