Refined geometric invariants and representation theory

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Enumerative Geometry
IHP workshop
March 2017

European Research Council
Established by the European Commission
FNSNF
Fonds national suisse
de la recherche scientifique
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Recollection on Frobenius algebras

- $\mathbb{K}$ field of characteristic 0
- *Frobenius algebra*: finite dimensional commutative unital $\mathbb{K}$-algebra F with non-degenerate pairing $\langle , \rangle$, which is symmetric $\langle a, bc \rangle = \langle ab, c \rangle$
- e.g. $\mathbb{K}R(\mathrm{G})$ for finite group G or $\mathbb{K}[\mathrm{G}]^{\mathrm{G}}$ with convolution
- 1+1D TQFT $\Leftrightarrow Z(S^1) = F$ with pairing $\langle , \rangle$
- (a_i) basis of orthogonal idempotents then

  $Z(\Sigma_g) = \sum_i \langle a_i, 1 \rangle^{1-g}$ *Verlinde formula*

- e.g. when $F = \mathbb{K}[\mathrm{G}]^{\mathrm{G}}$ we get $(\chi |G|^{-\frac{1}{2}})_{\chi \in \hat{\mathrm{G}}}$ orthogonal idempotents and

  $Z(\Sigma_g) = \sum_{\chi \in \hat{\mathrm{G}}} \left(\frac{|\mathrm{G}|^2}{\chi(1)^2} \right)^{g-1} = \frac{1}{|\mathrm{G}|} |Hom(\pi_1(\Sigma_g), \mathrm{G})|$

- 1+1D Chern-Simons theory with finite gauge group of (Freed–Quinn,1993)

- C genus g curve; fix group GL_n

$$\mathcal{M}^d_{\mathrm{Dol}} := \left\{ \begin{array}{c} \text{moduli space of semistable rank } n \\ \text{degree } d \text{ Higgs bundles } (E, \phi) \\ \phi \in H^0(C, End(E) \otimes K) \text{ Higgs field} \end{array} \right\}$$

$$\mathcal{M}^d_{\mathrm{B}} := \{ A_1, B_1, .., A_g, B_g \in \mathrm{GL}_n | \prod_{i=1}^{g} A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi i d}{n}} Id \} /\!\!/ \mathrm{PGL}_n$$

when $(d, n) = 1$ these are smooth non-compact varieties

- Non-Abelian Hodge Theorem: $\mathcal{M}^d_{\mathrm{Dol}} \overset{diff}{\cong} \mathcal{M}^d_{\mathrm{B}}$
(Hitchin, Donaldson, Corlette, Simpson)

Mixed Hodge polynomials

- (Deligne 1971) defines weight filtration $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H_c^k(X;\mathbb{Q})$ for any complex algebraic variety X
- $WH(X;q,t) = \sum \dim(W_i/W_{i-1}(H_c^k(X)))t^k q^{\frac{i}{2}}$, *mixed Hodge polynomial*
- $P_c(X;t) = WH(X;1,t)$, *Poincaré polynomial*
- $WH(X;q,-1)$, virtual weight polynomial of X

Theorem (Katz 2008)

If M is a smooth quasi-projective variety defined over $\mathbb{Z}$ and

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in q, then $WH(M(\mathbb{C});q,-1) = E(q)$.

Conjectures on $WH(\mathcal{M}_{\mathrm{B}}^d; q, t)$

- (Hausel-Villegas 2008) calculates $WH(\mathcal{M}_{\mathrm{B}}^d; q, -1) =$ $|\mathcal{M}_{\mathrm{B}}^d(\mathbb{F}_q)| = \sum_{\chi \in Irr(\mathrm{GL}_n(\mathbb{F}_q))} \frac{|\mathrm{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}} \frac{\chi(\xi_n^d)}{\chi(1)}$
- thus $WH(\mathcal{M}_{\mathrm{B}}^d; q, -1)$ computable from Frobenius algebra $\mathbb{C}[\mathrm{GL}_n(\mathbb{F}_q)]^{\mathrm{GL}_n(\mathbb{F}_q)}$

Conjecture (Hausel-Villegas, 2008)

$$\sum_\lambda \prod \frac{\left(z^{2l+1} - w^{2a+1}\right)^{2g}}{(z^{2l+2} - w^{2a})(z^{2l} - w^{2a+2})} T^{|\lambda|} = \exp\left(\sum_{n,k} \frac{WH\left(\mathcal{M}_{\mathrm{B}}^{n,d}; w^{2k}, -(zw)^{-2k}\right)}{(z^{2k}-1)(1-w^{2k})(zw)^{-d_n}} \frac{T^{nk}}{k} \right)$$

- checks:
  1. $z = 1/w$ and $n = 2$ (Hausel–Villegas, 2008)
  2. $w = 1$ (Chuang-Diaconescu-Pan 2010) from string theory arguments via (Mozgovoy 2011) and (Mellit 2016)
  3. $w = 1$? (Schiffmann 2015) computes $P_c(\mathcal{M}_{\mathrm{Dol}}^d; t)$
  4. $w = 1$ (Maulik–Pixton >2016) rigorizing (CDP 2010)
- Is there a Frobenius algebra and so representation theory behind $WH(\mathcal{M}_{\mathrm{B}}^d; q, t)$?

Conjectures on $PH(\mathcal{M}^d_{\mathrm{Dol}}; q, t)$

- (De Cataldo–Migliorini 2005) $\rightsquigarrow$ $P_0 \subset \cdots \subset P_k = H^k_c(\mathcal{M}^d_{\mathrm{Dol}}; \mathbb{Q})$ perverse filtration induced by the Hitchin map $\chi : \mathcal{M}^d_{\mathrm{Dol}} \to \mathcal{A}$
- $P(\mathcal{M}^d_{\mathrm{Dol}}; q, t) := \sum \dim(P_i/P_{i-1}(H^k_c(X)))t^k q^i$ perverse Hodge polynomial

Conjecture (De Cataldo–Hausel–Migliorini 2012)

$P_i(H^*_c(\mathcal{M}^d_{\mathrm{Dol}}) = W_{2i}(H^*_c(\mathcal{M}^d_{\mathrm{B}}))$ *in particular*
$PH(\mathcal{M}^d_{\mathrm{Dol}}; q, t) = WH(\mathcal{M}^d_{\mathrm{B}}; q, t)$

- checks:
  1. $n = 2$ (De Cataldo–Hausel–Migliorini 2012)
  2. (Chuang–Diaconescu–Pan 2013) string theoretical argument checking conjecture for $PH(\mathcal{M}^d_{\mathrm{Dol}}; q, t)$
- Is there a Frobenius algebra and so representation theory computing $PH(\mathcal{M}^d_{\mathrm{Dol}}; q, t)$?

- C smooth complex projective curve of genus g
- fix rank $n \in \mathbb{Z}_{>0}$, degree $d \in \mathbb{Z}$ and level $k \in \mathbb{Z}_{>0}$
- $\check{\mathcal{N}}_n^d$ moduli space of semi-stable rank n fixed degree d vector bundles on C; projective and smooth when $(d,n) = 1$
- $L \in \mathrm{Pic}(\check{\mathcal{N}}_n^d) \cong \mathbb{Z}$ ample generator of Picard group
- Verlinde formula (1988) for $\dim H^0(\check{\mathcal{N}}_n^d; L^k) = \chi(\check{\mathcal{N}}_n^d, L^k)$
- e.g. for $n = 2$ $d = 1$
  $\dim H^0(\check{\mathcal{N}}_2^1, L^k) = \sum_{j=1}^{2k+1}(-1)^{j+1}\left(\frac{k+1}{sin^2(\frac{j\pi}{2k+2})}\right)^{g-1} =$
  $\frac{1}{2}\underset{z=1}{\mathrm{Res}}\frac{(4k+4)^g}{\left(z^{k+1}-z^{-(k+1)}\right)((1-1/z)(1-z))^{g-1}}\frac{dz}{z}$
- proved for
  - $k=1$ by (Beauville–Narasimhan–Ramanan 1988)
  - $n = 2$ by (Szenes, Bertram–Szenes 1993)
  - $\vdots$
  - in all generality by (Teleman–Woodward, 2009)

Verlinde algebra

- Verlinde formula = partition function of a $1+1D$ TQFT
- $R(\mathrm{SU}_n)) \cong$ character ring of SU_n
- $R(\mathrm{SU}_n) \cong R(T_n)^{S_n} \cong (\mathbb{Z}[z_1, \ldots, z_n]/(z_1 \cdots z_n - 1))^{S^n}$
  irrep $\chi_\lambda \in Irr(\mathrm{SU}_n) \overset{BWB}{\rightsquigarrow} \chi(\mathcal{F}; L_\lambda) = s_\lambda \in R(T_n)^{S_n}$ Schur fn
  $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0) \in \mathbb{Z}^{n-1}$
- $Ver_n^k := \mathbb{C}R/(s_{(k+1)}, s_{(k+2)}, \ldots, s_{(k+n-1)})$ has basis $\{s_\lambda\}_{\lambda_1 \leq k}$
- declaring $\langle s_\lambda, s_{\eta^\dagger} \rangle = \delta_{\lambda\eta} \rightsquigarrow$ non-degenarate pairing

Theorem (Goodman-Wenzl 1990; Gepner 1991; Witten 1993)

$(Ver_n^k, \langle , \rangle)$ is a Frobenius algebra (i.e. $\langle a, bc \rangle = \langle ab, c \rangle$).
$\cong$ Verlinde algebra, with partition function giving Verlinde formulae

- the Frobenius algebra $(Ver_n^k, \langle , \rangle)$ arises as
  1. fusion algebra of level k representations of $\widehat{\mathfrak{sl}}_n$
  2. representation ring of $U_{e^{2\pi i(k+n)}}(\mathfrak{sl}_n)$
  3. representation ring of $H_n(e^{2\pi i(k+n)})$

Equivariant Verlinde formulae

- $\check{\mathcal{M}}_n^d \supset T^*\check{\mathcal{N}}_n^d$ moduli ss rank n fixed degree d Higgs bundles
- $\mathbb{T} := \mathbb{C}^\times$ acts on $\check{\mathcal{M}}_n^d$ by scaling Higgs field
- $L \in \mathrm{Pic}(\check{\mathcal{M}}_n^d)$ ample generator with $\mathbb{T}$ action trivial on $L^k|_{\check{\mathcal{N}}_n^d}$
- $\mathbb{T}$ acts on $H^0(\check{\mathcal{M}}_n^d, L^k)$ with weights ≥ 0
- $\mathrm{grdim}(H^0(\check{\mathcal{M}}_n^d, L^k)) = \sum_{i=0}^\infty \dim(H^0(\check{\mathcal{M}}_n^d, L^k)^i)t^i \in \mathbb{Z}[[t]]$
- $\mathrm{grdim}(H^0(\check{\mathcal{M}}_n^d, L^k)) = \chi_\mathbb{T}(\check{\mathcal{M}}_n^d, L^k) \in \widehat{K_\mathbb{T}(*)} \cong \widehat{R(\mathbb{T})} \cong \mathbb{Z}[[t^{\pm 1}]]$
- (Paradan 2011) $\rightsquigarrow$

  $\chi_\mathbb{T}(\check{\mathcal{M}}_n^d, L^k) = \sum_{F_i} \int_{F_i} ch_\mathbb{T}(L^k|_{F_i} \otimes \mathrm{Sym} N^* F_i) Todd(TF_i)$
- $F_i \subset (\check{\mathcal{M}}_n^d)^\mathbb{T}$ fixed point components
- (Hausel–Szenes, 2003) direct computation $\rightsquigarrow \chi_\mathbb{T}(\check{\mathcal{M}}_2^1, L^k) =$

$$\sum_{a=1,t,1/t} \operatorname*{Res}_{z=a} \frac{\frac{2^{2g-1}}{(1-t)^{g-1}}\left[k+1+\frac{zt}{1-zt}+\frac{t/z}{1-t/z}\right]^g}{\left[z^{k+1}\frac{1-t/z}{1-tz} - z^{-(k+1)}\frac{1-tz}{1-t/z}\right][(1-1/z)(1-z)(1-t/z)(1-tz)]^{g-1}} \frac{dz}{z},$$

- (Hausel–Szenes, 2003) conjecture for higher n
- recently (Halpern-Leistner 2016) and (Andersen–Gukov–Pei 2016) gave formulas for $\chi_\mathbb{T}(\mathcal{M}_G, L^k)$ for general G building on the work of (Teleman–Woodward 2009)

Equivariant Verlinde algebra for Higgs bundles

- (Gukov, Pei 2015) $\rightsquigarrow \chi_{\mathbb{T}}(\check{\mathcal{M}}_n^d, L^k)$ arises from a 1+1D TQFT
- $E_\lambda := \chi_{\mathbb{T}}(T^*\mathcal{F}; L_\lambda) = t_\lambda(t) P_\lambda / \psi_t \in R(T_n)^{S_n}[[t]]$

  $P_\lambda = \frac{\sum_{w\in S_n}(-1)^{\sigma(w)} z^{w(\lambda)} w(\Delta_t)}{\Delta_1 t_\lambda(t)} \in R(\mathrm{SU}_n)[t]$ Hall-Littlewood

  $\Delta_t = z^\rho \prod_{\alpha\in\Phi^-}(1-tz^\alpha)$; $\psi_t = \prod_{\alpha\in\Phi}(1-tz^\alpha)$; $t_\lambda(t) = \sum_{w\in St_{S_n}(\lambda)} t^{l(w)}$
- $\lambda_m := (k+1, 1, \dots, 1, 0, \dots, 0) = (k+1)\omega_1 + \omega_2 + \cdots + \omega_m$

  form $B_{\lambda_m} = \frac{\sum_{w\in S_n}(-1)^{\sigma(w)} z^{w(\lambda_m)}(1-tz^{w(-\theta)}) w(\Delta_t)}{\Delta_1} \in R(T_n)^{S_n}[t]$

  symmetric Bethe-Ansatz polynomial
- $QVer_n^k := \overline{\mathbb{C}(t)} R(T_n)^{S_n}[t]/(B_{\lambda_1}, \dots, B_{\lambda_{n-1}})$
- $\langle E_\lambda, E_{\eta^\dagger}\rangle_t := \delta_{\lambda\eta} \tilde{t}_\lambda(t)(1-t)^{n-1}$, $\tilde{t}_\lambda(t) = \sum_{w\in St_{\tilde{S}_n^k}(\lambda)} t^{l(w)}$

Theorem (Hausel–Szenes 2016)

$(E_\lambda)_{\lambda_1\le k}$ is a basis of $QVer_n^k$, and $(QVer_n^k, \langle,\rangle_t)$ is a Frobenius algebra computing equivariant Verlinde formula

- $(QVer_n^k, \langle,\rangle_t)_{t=0} \cong (Ver_n^k, \langle,\rangle)$
- is $(QVer_n^k, \langle,\rangle_t)$ the fusion algebra of some deformation of $\widehat{\mathfrak{sl}_n}$?

Summary

- we found that
- Frobenius algebra $\mathbb{C}[\mathrm{GL}_n(\mathbb{F}_q)]^{\mathrm{GL}_n(\mathbb{F}_q)} \rightsquigarrow PH(\mathcal{M}_{\mathrm{Dol}}; q, -1)$

Problem

Is there t-deformation of $\mathbb{C}[\mathrm{GL}_n(\mathbb{F}_q)]^{\mathrm{GL}_n(\mathbb{F}_q)}$ *computing* $PH(\mathcal{M}_{\mathrm{Dol}}; q, t)$*?*

- Frobenius $\mathbb{C}$-algebra $(Ver_n^k, \langle , \rangle) \cong R_k(\widehat{\mathfrak{sl}_n})$ computes $\chi(\check{\mathcal{N}}; L^k)$
- Frobenius $\overline{\mathbb{C}(t)}$-algebra $(QVer_n^k, \langle , \rangle_t)$ computes $\chi_{\mathbb{T}}(\check{\mathcal{M}}_{\mathrm{Dol}}; L^k)$

Problem

Is there a t-deformation of $R_k(\widehat{\mathfrak{sl}_n})$ *computing* $\chi_{\mathbb{T}}(\check{\mathcal{M}}_{\mathrm{Dol}}; L^k)$*?*