MATH 541: Abstract Algebra
Prof. Ram, Fall 2003

HOMEWORK 1
DUE September 8 and 15, 2003

Problem A. Understanding definitions

(1) Define operation, associative, commutative, identity and inverse.

(2) Define set, subset, and equal (for sets).

(3)
  (a) Define function, injection, surjection, and bijection.
  (b) Give an example of a function, and of something that looks like a function but is not a function.

(4)
  (a) Define function, injection, surjection, bijection and image.
  (b) Show that if $f\colon S \to T$ is a function then f is surjective if and only if $\mathrm{im} f = T$.

(5) Define inverse function and show that a function f is a bijection if and only if the inverse function to f exists.

(6) Define group, subgroup and multiplication table. Why isn't $\{0, 1, 2, 3, 4, 5\}$ a group?

(7)
  (a) Define center, abelian group, and finite group.
  (b) Show that the center of a group G is a subgroup of G.
  (c) Let G be a group. Show that G is abelian if and only if $G = Z(G)$.

(8) Show that the identity of a group is unique.

(9) Show that the inverse of an element g in a group G is unique.

(10) Explain why $0 + 0 = 0$, $1 \cdot 1 = 1$, $-(-5) = 5$ and $1/(1/3) = 3$, and show that if g_1, g_2 are elements of a group G, then $(g_1g_2)^{-1} = g_2^{-1}g_1^{-1}$.

(11)
  (a) Define the order of a group and the order of an element of a group.
  (b) Give an example of a group, and list the order of the group and the orders of all its elements.

(12) Let G be a group and let M be the multiplication table of G. Let $g \in G$. Show that each row and each column of M contains g exactly once.

(13) Let H_1 and H_2 be subgroups of a group G.
  (a) Show that $H_1 \cap H_2$ is a subgroup of G.
  (b) Show that $H_1 \cup H_2$ is a subgroup of G if and only if $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

(14)
  (a) Define the direct product of groups.
  (b) Let H_1 and H_2 be subgroups of groups G_1 and G_2, respectively. Show that $H_1 \times H_2$ is a subgroup of $G_1 \times G_2$.

(15) Let G be a group and let S be a subset of G. Show that the subgroup generated by S is unique (if it exists).

(16) Let G be a group and let S be a subset of G. Show that the subgroup generated by S exists.

(17)
  (a) Define group homomorphism.
  (b) Show that if $\phi: G \to H$ is a group homomorphism then $f(1_G) = 1_H$, where 1_G is the identity element of G and 1_H is the identity element of H.
  (c) Show that if $\phi: G \to H$ is a group homomorphism and $g \in G$ then $f(g)^{-1} = f(g^{-1})$.

(18)
  (a) Define group homomorphism, isomorphism, kernel and image.
  (b) Show that the kernel and the image of of a group homomorphism are subgroups.

(19)
  (a) Define group homomorphism, isomorphism, kernel and image.
  (b) Show that a group homomorphism ϕ is injective if and only if $\ker \phi = \{1\}$.
  (c) Show that a group homomorphism $\phi: G_1 \to G_2$ is surjective if and only if $\mathrm{im}\phi = G_2$.
  (d) Show that a group homomorphism is an isomorphism if and only if it is both injective and surjective.

(20) Show that, in the exact sequence

$$\{1\} \longrightarrow K \xrightarrow{g} G \xrightarrow{f} H \longrightarrow \{1\}$$

the homomorphism g is always injective and the homomorphism f is always surjective.

(21)

(a) Let H be a subgroup of a group G. The **canonical injection** is the map $\iota\colon H \to G$ given by

$$\begin{array}{rccc} \iota: & H & \longrightarrow & G \\ & h & \longmapsto & h\,. \end{array}$$

Show that ι is a well defined injective group homomorphism.

(b) Let $f\colon G \to H$ be a group homomorphism. Show that

$$\{1\} \longrightarrow \ker f \xrightarrow{\iota} G \xrightarrow{f} \mathrm{im} f \longrightarrow \{1\}$$

is an exact sequence.

(22) Let $f\colon G \to H$ be a group homomorphism. Let M be a subgroup of G and define

$$f(M) = \{f(m) \mid m \in M\}.$$

(a) Show that $f(M)$ is a subgroup of H.
(b) Show that $f(M) \subseteq \mathrm{im} f = f(G)$.

(23) Let $f\colon G \to H$ be a group homomorphism. Let N be a subgroup of H and define

$$f^{-1}(N) = \{g \in G \mid f(g) \in N\}.$$

(a) Show that $f^{-1}(N)$ is a subgroup of G.
(b) Show that $f^{-1}(N) \supseteq \ker f = f^{-1}(1)$.

(24) Let $f\colon G \to H$ be a group homomorphism.
(a) Let M be a subgroup of G and show that $M \subseteq f^{-1}(f(M))$.
(b) Give an example of a homomorphism $f\colon G \to H$ and a subgroup m of G such that $M \neq f^{-1}(f(M))$.
(c) Show that if M is a subgroup of G that contains $\ker f$ then $M = f^{-1}(f(M))$.

(25) Let $f\colon G \to H$ be a group homomorphism.
(a) Let N be a subgroup of H and show that $f(f^{-1}(N)) \subseteq N$.
(b) Give an example of a homomorphism $f\colon G \to H$ and a subgroup N of H such that $N \neq f(f^{-1}(N))$.
(c) Show that if N is a subgroup of H and $N \subseteq \mathrm{im} f$ then $N = f(f^{-1}(N))$.

(26) Let $f\colon G \to H$ be a group homomorphism. Show that there is a bijection between subgroups of G that contain $\ker f$ and subgroups of H that are contained in $\mathrm{im} f$.

(27) Define relation, transitive, symmetric and reflexive.

(28) Define partial order, total order and well ordering.

(29) Define Hasse diagram and lattice.

(30)
   (a) Define equivalence relation, equivalence classes partition (of a set) and partition (of a positive integer n).
   (b) Show that the equivalence classes of an equivalence relation form a partition.

Problem B. Examples of groups

(1) Define $\mathbb{Z}$ and $\mathbb{Z}/\ell\mathbb{Z}$.

(2) Define the Klein 4-group, the symmetric group S_3 and the quaternion group.

(3) Define the cyclic groups and the dihedral groups.

(4) Define the symmetric groups and the alternating groups.

(5) Define the general linear groups, the special linear groups, the orthogonal groups, the special orthgonal groups, the symplectic groups, the unitary groups, and the special unitary groups.

(6) Define the groups $G_{r,1,n}$ and the groups $G_{r,p,n}$.

(7) Define the tetrahedral group, the octahedral group, and the icosahedral group.

(8) Find all groups of order ≤ 4 by determining their multiplication tables.

(9) Compute the multiplication tables of $\mathbb{Z}$ and $\mathbb{Z}/\ell\mathbb{Z}$, where ℓ is a positive integer.

(10) Write out the multiplication tables for $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ and for S_3.

(11) Give an example of an abelian group and a nonabelian group.

(12) Give an example of an infinite cyclic group and a finite cyclic group.

(13) Give two examples of finite dihedral groups. Be sure to show that your examples are different.

(14) Give three examples of abelian groups of order 8. Be sure to show that your examples are all different.

(15) Show that the quaternion group is not isomorphic to the dihedral group of order 8.

(16) Show that $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ and S_3 are dihedral groups.

(17) Let G be a cyclic group. Show that

$$G \cong \begin{cases} \mathbb{Z}, & \text{if } G \text{ is infinite,} \\ \mathbb{Z}/\ell\mathbb{Z}, & \text{if } |G| = \ell. \end{cases}$$

(18) Let G_1 and G_2 be finite dihedral groups. Show that $G_1 \cong G_2$ if and only if G_1 and G_2 have the same order.

(19) Show that any two infinite dihedral groups are isomorphic.

(20)
 (a) Let $i = \sqrt{-1}$ in the complex numbers $\mathbb{C} = \{a + bi \mid a \in \mathbb{R}, b \in \mathbb{R}\}$. Show that $\langle i \rangle \cong \mathbb{Z}/4\mathbb{Z}$.
 (a) Show that every nonzero element of $\mathbb{C}$ is invertible.
 (b) Show that every nonzero element of $\mathbb{H}$ is invertible.
 (c) Show that $\mathbb{C}$ is commutative and $\mathbb{H}$ is not commutative.

(21) Find the orders of the Klein 4-group, the symmetric group S_3 and the quaternion group.

(22) Find the orders of the cyclic groups and the dihedral groups.

(23) Find the orders of the symmetric groups and the alternating groups.

(24) Find the orders of the tetrahedral, octahedral, and icosahedral groups.

(25) Find the orders of the groups $G_{r,1,n}$ and $G_{r,p,n}$.

(26) Show that $G_{r,1,1}$ is a cyclic group. Which one is it?

(27) Show that $G_{r,r,2}$ is a dihedral group. Which one is it?

(28) Show that $G_{1,1,n}$ is the symmetric group. Which one is it?

(29) Show that $G_{2,1,3}$ is the octahedral group.

(30) Find the centers of the cyclic groups and the dihedral groups.

(31) Find the centers of the symmetric groups and the alternating groups.

(32) Find the centers of the groups $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$.

(33) Find the centers of the groups $G_{r,1,n}$ and $G_{r,p,n}$.

(34) Show that the determinant

$$\begin{array}{rccc} \det\colon & GL_n(\mathbb{C}) & \longrightarrow & GL_1(\mathbb{C}) \\ & g & \longmapsto & \det(g) \end{array}$$

is a group homomorphism.

(35)
(a) Show that $\{z \in \mathbb{C} \mid z^n = 1\}$ is a cyclic group.
(b) Draw the group $\{z \in \mathbb{C} \mid z^5\}$ in the complex plane.

(36) Draw the sugroup lattices for the groups $\mathbb{Z}/28\mathbb{Z}$, $\mathbb{Z}/29\mathbb{Z}$, $\mathbb{Z}/30\mathbb{Z}$.

(37) Draw the sugroup lattices for the groups $\mathbb{Z}/25\mathbb{Z}$, $\mathbb{Z}/26\mathbb{Z}$, $\mathbb{Z}/27\mathbb{Z}$.

(38) Draw the sugroup lattices for the dihedral groups of orders 14, 15, 16, 18, and 20.

(39) Draw the sugroup lattices for the symmetric groups S_3 and S_4.

(40) Draw the sugroup lattices of the tetrahedral group and the octahedral group.

(41) Draw the sugroup lattice of the icosahedral group.