MATH 541: Abstract Algebra
Prof. Ram, Fall 2003

HOMEWORK 3
DUE October 14, 2003

Problem A. Understanding definitions

(1) Define conjugate, conjugacy class, and centralizer and normalizer of an element and explain what these have to do with actions.

(2) Let G be a group.
   (a) Define the normalizer of a subset of G.
   (b) Explain how the centralizer of a subset of G relates to group actions.
   (c) Let H be a subgroup of G. Show that the normalizer of H is the largest subgroup of G which contains H as a normal subgroup.

(3) Let G be a group.
   (a) Define the centralizer of a subset of G.
   (b) Explain how the centralizer of a subset of G relates to group actions.

(4) Let G be a group. If S is a subset of G let $Z_G(S)$ denote the centralizer of S in G. If $g \in G$ let $Z_G(g)$ denote the centralizer of g and let $\mathcal{C}_g$ denote the conjugacy class of g.
   (a) Show that $g \in Z(G)$ if and only if $Z_G(g) = G$.
   (b) Show that $g \in Z(G)$ if and only if $\mathcal{C}_g = \{g\}$.
   (c) Show that $Z_G(G) = Z(G)$, where $Z(G)$ is the center of G.

(5) State and prove the class equation.

Problem B. Examples of groups

(1) Let ℓ be a positive integer.
   (a) Find the orders, centralizers, and conjugacy classes of the elements in the group $\mathbb{Z}/\ell\mathbb{Z}$.
   (b) Find the orders, centralizers, and conjugacy classes of the elements in the group $\mathbb{Z}$.

(2) For r equal to 2,3,4,5 and 6, find the orders, centralizers, and conjugacy classes of the elements of the group $I_2(r)$.

(3) Let r be a positive integer. Describe the orders, centralizers, and conjugacy classes of the elements of the group $I_2(r)$.

(4) For $n = 1, 2, 3, 4, 5$ find the orders, centralizers, and conjugacy classes of the elements of the group S_n.

(5) For $n = 1, 2, 3, 4, 5$ find the orders, centralizers, and conjugacy classes of the elements of the group A_n.

(6) Let n be a positive integer. Describe the orders, centralizers, and conjugacy classes of the elements of the group S_n.

(7) Let n be a positive integer. Describe the orders, centralizers, and conjugacy classes of the elements of the group A_n.

(8)
    (a) Describe the orders, centralizers, and conjugacy classes of the elements of the group $G_{2,1,1}$.
    (b) Describe the orders, centralizers, and conjugacy classes of the elements of the group $G_{2,1,2}$.
    (c) Describe the orders, centralizers, and conjugacy classes of the elements of the group $G_{2,1,3}$.

(9)
    (a) Describe the orders, centralizers, and conjugacy classes of the elements of the group $G_{2,1,4}$.
    (b) Describe the orders, centralizers, and conjugacy classes of the elements of the group $G_{2,1,n}$.

(10) Let r and n be positive integers. Describe the orders, centralizers, and conjugacy classes of the elements of the group $G_{r,1,n}$.

(11) Let n be a positive integer.
    (a) Describe the orders, centralizers, and conjugacy classes of the elements of the group $G_{2,1,n}$.
    (b) Describe the orders, centralizers, and conjugacy classes of the elements of the group $G_{2,2,n}$.

(12) Let r, p and n be positive integers such that p divides r. Describe the orders, centralizers, and conjugacy classes of the elements of the group $G_{r,p,n}$.

(13) Describe the orders, centralizers, and conjugacy classes of the elements of the tetrahedral group.

(14) Describe the orders, centralizers, and conjugacy classes of the elements of the octahedral group.

(15) Describe the orders, centralizers, and conjugacy classes of the elements of the icosahedral group.

(16) Describe the orders, centralizers, and conjugacy classes of the elements of the group $GL_n(\mathbb{C})$.

(17) Analyze the class equation for the groups $\mathbb{Z}$ and $\mathbb{Z}/\ell\mathbb{Z}$, where ℓ is a positive integer.

(18) Analyze the class equation for the groups $I_2(r)$.

(19) Analyze the class equation for the groups $G_{2,1,1}$, $G_{2,1,2}$, $G_{2,1,3}$, $G_{2,1,4}$, and $G_{2,1,n}$.

(20) Analyze the class equation for the groups $G_{r,1,n}$.

(21) Analyze the class equation for the groups $G_{2,2,3}$, $G_{2,2,n}$, and $G_{r,p,n}$.

(22) Analyze the class equation for the groups S_3, S_4, S_5 and S_n,

(23) Analyze the class equation for the groups A_3, A_4, A_5, and A_n.

(24) Analyze the class equation for the tetrahedral group.

(25) Analyze the class equation for the octahedral group.

(26) Analyze the class equation for the icosahedral group.

(27) Analyze the class equation for the group $GL_n(\mathbb{C})$.