§4T. Alternating group

(1.4.1) Definition.

- The **alternating group** A_n is the subgroup of even permutations of S_n.

The alternating group A_n is the kernel of the sign homomorphism of the symmetric group;

$$A_n = \ker(\varepsilon), \qquad \text{where} \qquad \begin{array}{rccc} \varepsilon\colon & S_n & \to & \{\pm 1\} \\ & \sigma & \mapsto & \det(\sigma). \end{array}$$

HW: Show that A_n is a normal subgroup of S_n.
HW: Show that $|A_n| = n!/2$.

Conjugacy classes

Since A_n is a normal subgroup of S_n, A_n is a union of conjugacy classes of S_n. Let $\mathcal{C}_\lambda$ be a conjugacy class of S_n corresponding to a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Then the following Proposition says:

1) The conjugacy class $\mathcal{C}_\lambda$ is contained in A_n if an even number of the λ_i are even numbers.
2) If the parts λ_i of λ are all odd and are all distinct then $\mathcal{C}_\lambda$ is a union of two conjugacy classes of A_n and these two conjugacy classes have the same size.
3) Otherwise $\mathcal{C}_\lambda$ is also a conjugacy class of A_n.

(1.4.2) Proposition. *Suppose that $\sigma \in A_n$. Let $\mathcal{C}_\sigma$ denote the conjugacy class of σ in S_n and let $\mathcal{A}_\sigma$ denote the conjugacy class of σ in A_n.*

a) Then σ has an even number of cycles of even length.

b)

$$|\mathcal{A}_\sigma| = \begin{cases} \dfrac{|\mathcal{C}_\sigma|}{2}, & \textit{if all cycles } \sigma \textit{ are of different odd lengths;} \\ |\mathcal{C}_\sigma|, & \textit{otherwise;} \end{cases}$$

The proof of Proposition (1.4.2) uses the following lemma.

(1.4.3) Lemma. *Let $\sigma \in A_n$ and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be the cycle type of σ. Let γ_λ be the permutation given, in cycle notation, by*

$$\gamma_\lambda = (1, 2, \cdots, \lambda_1)(\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + 1, \cdots) \cdots .$$

Let S_σ denote the stabilizer of σ under the action of S_n on itself by conjugation. Then,

a) $S_\sigma \subseteq A_n$ if and only if $S_{\gamma_\lambda} \subseteq A_n$.

b) $S_{\gamma_\lambda} \subseteq A_n$ if and only if γ_λ has all odd cycles of different lengths.

A_n is simple, $n \neq 4$.

A group is simple if it has no nontrivial normal subgroups. The trivial normal subgroups are the whole group and the subgroup containing only the identity element.

(1.4.4) Theorem.

a) If $n \neq 4$ then A_n is simple.

b) The alternating group A_4 has a single nontrivial proper normal subgroup given by

$$N = \{(1234), (2143), (3412), (4321)\},$$

where the permutations are represented in one-line notation.

The proof of Theorem (1.4.4) uses the following lemma.

(1.4.5) Lemma. *Suppose N is a normal subgroup of A_n, $n > 4$, and N contains a 3-cycle. Then $N = A_n$.*