§2T. The dihedral groups D_n, $n \geq 2$

(0.2.1) Definition.

- The **dihedral group**, D_n, is the set $D_n = \{1, x, x^2, \ldots, x^{n-1}, y, xy, x^2y, \ldots, x^{n-1}y\}$ with the operation given by

$$(x^i y^j)(x^k y^l) = x^{(i+k) \bmod n} y^{(j+l) \bmod 2}.$$

HW: Show that the order of the dihedral group D_n is $2n$.

(0.2.2) Proposition. *The orders of the elements in the dihedral group D_n are*

$$o(1) = 1, \qquad o(x^k) = \gcd(k, n), \qquad o(x^k y) = 2, \qquad 0 < k \leq n-1.$$

Conjugacy classes, normal subgroups, and the center

(0.2.3) Proposition.

a) The conjugacy classes of the dihedral group D_2 are the sets

$$\mathcal{C}_1 = \{1\}, \qquad \mathcal{C}_x = \{x\}, \qquad \mathcal{C}_y = \{y\}, \qquad \mathcal{C}_{xy} = \{xy\}.$$

b) If n is even and $n \neq 2$, then the conjugacy classes of the dihedral group D_n are the sets

$$\begin{aligned} \mathcal{C}_1 = \{1\}, \qquad \mathcal{C}_{x^{n/2}} = \{x^{n/2}\}, \qquad & \mathcal{C}_{x^k} = \{x^k, x^{-k}\}, \quad 0 < k < n/2, \\ \mathcal{C}_y = \{y, x^2y, x^4y, \ldots, x^{n-2}y\}, \qquad & \mathcal{C}_{xy} = \{xy, x^3y, x^5y, \ldots, x^{n-1}y\} \end{aligned}$$

c) If n is odd then the conjugacy classes of the dihedral group D_n are the sets

$$\begin{aligned} \mathcal{C}_1 = \{1\}, \qquad \mathcal{C}_{x^k} = \{x^k, x^{-k}\}, \quad 0 < k < n/2, \\ \mathcal{C}_y = \{y, xy, x^2y, x^3y, \ldots, x^{n-1}y\}. \end{aligned}$$

(0.2.4) Proposition. *Let $\langle a, b, \cdots \rangle$ denote the subgroup generated by elements $a, b, \ldots$.*

a) The normal subgroups of the dihedral group D_2 are the subgroups

$$\langle x \rangle, \qquad \langle y \rangle, \qquad \langle xy \rangle.$$

b) If n is even and $n \neq 2$ then the normal subgroups of the dihedral group D_n are the subgroups

$$\langle x^k \rangle, \qquad 0 \leq k \leq n-1, \qquad \langle x^2, y \rangle, \qquad \langle x^2, xy \rangle.$$

c) If n is odd then the normal subgroups of the dihedral group D_n are the subgroups

$$\langle x^k \rangle, \qquad 1 \leq k \leq n-1.$$

(0.2.5) Proposition.

a) The center of the dihedral group D_2 is the subgroup $Z(D_2) = D_2$.

b) If n is even, $n \neq 2$, then the center of the dihedral group D_n is the subgroup $Z(D_n) = \{1, x^{n/2}\}$.

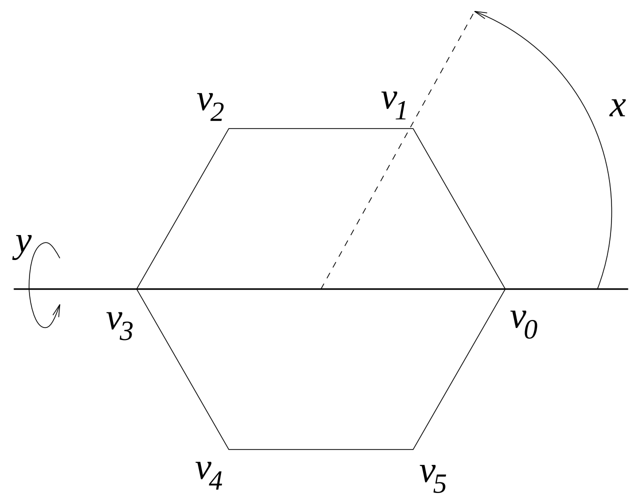
c) If n is odd, then the center of the dihedral group D_n is the subgroup $Z(D_n) = (1)$.

The action of D_n on an n-gon

(0.2.6) Proposition. *Let F be an n-gon with vertices $v_0, v_1, \ldots, v_{n-1}$ numbered counterclockwise around F. Then there is an action of the group D_n on the n-gon F such that*

x acts by rotating the n-gon by an angle of $2\pi/n$;

y acts by reflecting about the line which contains the vertex v_0 and the center of F.



Generators and relations

(0.2.7) Proposition.

a) The dihedral group $D_n = \{1, x, x^2, \ldots, x^{n-1}, y, xy, x^2y, \ldots, x^{n-1}y\}$ is generated by the elements x and y.

(b) The elements x and y in D_n satisfy the relations

$$x^n = 1, \qquad y^2 = 1, \qquad yx = x^{-1}y.$$

(0.2.8) Theorem. *The dihedral group D_n has a presentation by generators and relations by*

$$D_n = \langle x, y \mid x^n = 1, \;\; y^2 = 1, \;\; yx = x^{-1}y \rangle.$$