§1T. The cyclic group $\mathbb{Z}_2$ of order 2

There are at least two natural ways of defining the group $\mathbb{Z}_2$. The isomorphism which shows that these two definitions are the same is given in the rightmost column of the following table.

Set	Operation	Multiplication Table	Isomorphism
$\mathcal{Z}_2 = \{1, -1\} = \{\pm 1\}$	ordinary multiplication of integers	$\begin{array}{c\|cc} \times & 1 & -1 \\ \hline 1 & 1 & -1 \\ -1 & -1 & 1 \end{array}$	$\begin{array}{cccc} \varphi: & \mathbb{Z}_2 & \to & \mathcal{Z}_2 \\ & 0 & \mapsto & 1 \\ & 1 & \mapsto & -1 \end{array}$
$\mathbb{Z}_2 = \{0, 1\}$	addition modulo 2	$\begin{array}{c\|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$	

Center	Abelian	Conjugacy classes	Subgroups
$Z(\mathbb{Z}_2) = \mathbb{Z}_2$	Yes	$\mathcal{C}_1 = \{1\}$ $\mathcal{C}_{-1} = \{-1\}$	$H_0 = \mathbb{Z}_2$ $H_1 = (1)$

Elements

Element g	Order $o(g)$	Centralizer Z_g
1	1	$\mathbb{Z}_2$
-1	2	$\mathbb{Z}_2$

Generators and relations

Generators	Relations
g	$g^2 = 1$

Some Homomorphisms

Homomorphism	Kernel	Image
$\begin{array}{cccc} \phi_0: & \mathbb{Z}_2 & \to & (1) \\ & 1 & \mapsto & 1 \\ & -1 & \mapsto & 1 \end{array}$	$\ker \phi_0 = \mathbb{Z}_2$	$\operatorname{im} \phi_0 = (1)$
$\begin{array}{cccc} \phi_1: & \mathbb{Z}_2 & \to & \mathbb{Z}_2 \\ & 1 & \mapsto & 1 \\ & -1 & \mapsto & -1 \end{array}$	$\ker \phi_1 = (1)$	$\operatorname{im} \phi_1 = \mathbb{Z}_2$

Subgroups

Subgroup Lattice

Orders	**Inclusions**
2	$\mathbb{Z}_2$
1	$\{(1,1)\}$

Subgroups H_i	Structure	Order $\lvert H_i \rvert$	Index	Normal	Quotient Group
$H_0 = \mathbb{Z}_2$	$H_0 = \mathbb{Z}_2$	2	$[\mathbb{Z}_2 : \mathbb{Z}_2] = 1$	Yes	$\mathbb{Z}_2/H_0 \simeq (1)$
$H_1 = (1)$	$H_1 = (1)$	1	$[\mathbb{Z}_2 : (1)] = 2$	Yes	$\mathbb{Z}_2/(1) \simeq \mathbb{Z}_2$

Subgroup H_i	Normalizer N_{H_i}	Centralizer Z_{H_i}
$H_0 = \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$H_1 = (1)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Subgroups	Cosets	Right Cosets
$H_0 = \mathbb{Z}_2$	$\mathbb{Z}_2 = \{1, -1\}$	$\mathbb{Z}_2 = \{1, -1\}$
$H_1 = (1)$	$H_1 = \{1\}$ $(-1)H_1 = \{-1\}$	$H_1 = \{1\}$ $H_1(-1) = \{-1\}$

§2T. The Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$

Let us make some shorter notations for the following matrices.

$$(1,1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (-1,1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (1,-1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (-1,-1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Klein 4-group is the group of order 4 defined as in the following table.

Set	Operation
$\mathbb{Z}_2 \times \mathbb{Z}_2 = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}$	ordinary matrix multiplication

The complete multiplication table for this group is as follows.

Multiplication Table

	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
$(1,1)$	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
$(1,-1)$	$(-1,1)$	$(1,1)$	$(-1,-1)$	$(1,-1)$
$(-1,1)$	$(1,-1)$	$(-1,-1)$	$(1,1)$	$(-1,1)$
$(-1,-1)$	$(-1,-1)$	$(1,-1)$	$(-1,1)$	$(1,1)$

HW: Show that this group, as defined above, is isomorphic to the direct product of a cyclic group of order 2, $\mathbb{Z}_2$, with another cyclic group of order 2, $\mathbb{Z}_2$.

Center	Abelian	Conjugacy Classes	Subgroups
$Z(G) = \mathbb{Z}_2 \times \mathbb{Z}_2$	Yes	$\mathcal{C}_{(1,1)} = \{(1,1)\}$	$H_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$
		$\mathcal{C}_{(1,-1)} = \{(1,-1)\}$	$H_1 = \{(1,1),(1,-1)\}$
		$\mathcal{C}_{(1,-1)} = \{(1,-1)\}$	$H_3 = \{(1,1),(-1,-1)\}$
		$\mathcal{C}_{(1,-1)} = \{(1,-1)\}$	$H_4 = \{(1,1)\}$

Elements

Element g	Order $o(g)$	Centralizer Z_g	Conjugacy Class $\mathcal{C}_g$
$(1,1)$	1	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathcal{C}_{(1,1)}$
$(1,-1)$	2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathcal{C}_{(1,-1)}$
$(-1,1)$	2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathcal{C}_{(-1,1)}$
$(-1,-1)$	2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathcal{C}_{(-1,-1)}$

Subgroups

Subgroups H_i	Structure	Order $\|H_i\|$	Index	Normal	Quotient Group
$H_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	4	1	Yes	$(\mathbb{Z}_2 \times \mathbb{Z}_2)/H_0 \simeq (1)$
$H_1 = \{(1,1),(1,-1)\}$	$\mathbb{Z}_2$	2	2	Yes	$(\mathbb{Z}_2 \times \mathbb{Z}_2)/H_1 \simeq \mathbb{Z}_2$
$H_2 = \{(1,1),(-1,1)\}$	$\mathbb{Z}_2$	2	2	Yes	$(\mathbb{Z}_2 \times \mathbb{Z}_2)/H_2 \simeq \mathbb{Z}_2$
$H_3 = \{(1,1),(-1,-1)\}$	$\mathbb{Z}_2$	2	2	Yes	$(\mathbb{Z}_2 \times \mathbb{Z}_2)/H_3 \simeq \mathbb{Z}_2$
$H_4 = \{(1,1)\}$	(1)	1	4	Yes	$(\mathbb{Z}_2 \times \mathbb{Z}_2)/H_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$

Subgroup Lattice

Orders	Inclusions		
4		$\mathbb{Z}_2 \times \mathbb{Z}_2$	
2	$\{(1,1),(1,-1)\}$	$\{(1,1),(-1,1)\}$	$\{(1,1),(-1,-1)\}$
1		$\{(1,1)\}$	

Subgroups	Cosets	Right Cosets
H_0	$H_0 = \{(\pm1,\pm1)\}$	$H_0 = \{\pm1,\pm1\}$
H_1	$H_1 = \{(1,1),(1,-1)\}$ $(-1,1)H_1 = \{(-1,1),(-1,-1)\}$	$H_1 = \{(1,1),(1,-1)\}$ $H_1(-1,1) = \{(-1,1),(-1,-1)\}$
H_2	$H_2 = \{(1,1),(-1,1)\}$ $(1,-1)H_2 = \{(1,-1),(-1,-1)\}$	$H_2 = \{(1,1),(-1,1)\}$ $H_2(1,-1) = \{(1,-1),(-1,-1)\}$
H_3	$H_3 = \{(1,1),(-1,-1)\}$ $(1,-1)H_3 = \{(1,-1),(-1,1)\}$	$H_3 = \{(1,1),(-1,-1)\}$ $H_3(1,-1) = \{(1,-1),(-1,1)\}$
H_4	$H_4 = \{(1,1)\}$ $(-1,1)H_4 = \{(-1,1)\}$ $(1,-1)H_4 = \{(1,-1)\}$ $(-1,-1)H_4 = \{(-1,-1)\}$	$H_4 = \{(1,1)\}$ $H_4(-1,1) = \{(-1,1)\}$ $H_4(1,-1) = \{(1,-1)\}$ $H_4(-1,-1) = \{(-1,-1)\}$

Subgroup H_i	Normalizer N_{H_i}	Centralizer Z_{H_i}
H_0	H_0	H_0
H_1	H_0	H_0
H_2	H_0	H_0
H_3	H_0	H_0
H_4	H_0	H_0

Generators and relations

Generators	Relations
x, y	$x^2 = 1$
	$y^2 = 1$
	$xy = yx$

Some Homomorphisms

Homomorphism	Kernel	Image
$\phi_0\colon \begin{array}{rcl} \mathbb{Z}_2 \times \mathbb{Z}_2 & \to & (1) \\ (-1,1) & \mapsto & 1 \\ (1,-1) & \mapsto & 1 \end{array}$	$\ker \phi_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$	$\operatorname{im} \phi_0 = (1)$
$\phi_1\colon \begin{array}{rcl} \mathbb{Z}_2 \times \mathbb{Z}_2 & \to & \mathbb{Z}_2 \\ (-1,1) & \mapsto & -1 \\ (1,-1) & \mapsto & 1 \end{array}$	$\ker \phi_1 = H_1$	$\operatorname{im} \phi_1 = \mathbb{Z}_2$
$\phi_1\colon \begin{array}{rcl} \mathbb{Z}_2 \times \mathbb{Z}_2 & \to & \mathbb{Z}_2 \\ (-1,1) & \mapsto & 1 \\ (1,-1) & \mapsto & -1 \end{array}$	$\ker \phi_1 = H_2$	$\operatorname{im} \phi_1 = \mathbb{Z}_2$
$\phi_1\colon \begin{array}{rcl} \mathbb{Z}_2 \times \mathbb{Z}_2 & \to & \mathbb{Z}_2 \\ (-1,1) & \mapsto & -1 \\ (1,-1) & \mapsto & -1 \end{array}$	$\ker \phi_1 = H_3$	$\operatorname{im} \phi_1 = \mathbb{Z}_2$

§3T. $S_3 \simeq D_3$: the nonabelian group of order 6

Let

$$1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$(13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad (132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad (123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The groups S_3 and D_3 are as in the following table.

Set	Operation
$S_3 = \{1, (12), (23), (13), (132), (123)\}$	ordinary matrix multiplication
$D_3 = \{1, x, x^2, y, xy, x^2y\}$	$x^iy^jx^ky^l = x^{(i-k) \bmod 3}y^{(j+l) \bmod 2}$

The complete multiplication tables for these groups are as follows.

Multiplication Tables

S_3	1	(12)	(23)	(13)	(132)	(123)
1	1	(12)	(23)	(13)	(132)	(123)
(12)	(12)	1	(123)	(132)	(13)	(23)
(23)	(23)	(132)	1	(123)	(12)	(13)
(13)	(13)	(123)	(132)	1	(23)	(12)
(132)	(132)	(23)	(13)	(12)	(123)	1
(123)	(123)	(13)	(12)	(23)	1	(132)

D_3	1	y	x^2y	xy	x^2	x
1	1	y	x^2y	xy	x^2	x
y	y	1	x	x^2	xy	x^2y
x^2y	x^2y	x^2	1	x	y	xy
xy	xy	x	x^2	1	x^2y	y
x^2	x^2	x^2y	xy	y	x	1
x	x	xy	y	x^2y	1	x^2

HW: Prove that the group homorphism given as in the following table is an isomorphism.

Isomorphism

$$\begin{array}{rrcl} \Phi: & D_3 & \to & S_3 \\ & x & \mapsto & (123) \\ & y & \mapsto & (12) \end{array}$$

Center	Abelian	Conjugacy Classes	Subgroups
$Z(S_3) = (1)$	No	$\mathcal{C}_{(1^3)} = \{1\}$	$H_0 = S_3$
		$\mathcal{C}_{(21)} = \{(12), (23), (13)\}$	$H_1 = \{1, (132), (123)\}$
		$\mathcal{C}_{(3)} = \{(123), (132)\}$	$H_2 = \{1, (12)\}$
			$H_3 = \{1, (13)\}$
			$H_4 = \{1, (23)\}$
			$H_5 = \{1\}$

Elements

Element g	Order $o(g)$	Centralizer Z_g	Conjugacy Class $\mathcal{C}_g$
1	1	S_3	$\mathcal{C}_{(1^3)}$
(12)	2	H_2	$\mathcal{C}_{(21)}$
(23)	2	H_4	$\mathcal{C}_{(21)}$
(13)	2	H_3	$\mathcal{C}_{(21)}$
(132)	3	H_1	$\mathcal{C}_{(3)}$
(123)	3	H_1	$\mathcal{C}_{(3)}$

Subgroups

Subgroup Lattice

Orders	Inclusions
6	S_3
3	$\{1, (123), (132)\}$
2	$\{1, (12)\}$ $\{1, (23)\}$ $\{1, (13)\}$
1	$\{1\}$

Subgroups	Structure	Index	Normal	Quotient Group
$H_0 = S_3$	$H_0 = S_3$	$[S_3 : S_3] = 1$	Yes	$S_3/H_0 \simeq (1)$
$H_1 = \{1, (132), (123)\}$	$H_1 \simeq Z_3 \simeq A_3$	$[S_3 : H_1] = 2$	Yes	$S_3/H_1 \simeq Z_2$
$H_2 = \{1, (12)\}$	$H_2 \simeq Z_2$	$[S_3 : H_2] = 3$	No	
$H_3 = \{1, (13)\}$	$H_3 \simeq Z_2$	$[S_3 : H_3] = 3$	No	
$H_4 = \{1, (23)\}$	$H_4 \simeq Z_2$	$[S_3 : H_4] = 3$	No	
$H_5 = \{1\}$	$H_5 = \{1\}$	$[S_3 : H_5] = 6$	Yes	$S_3/(1) \simeq S_3$

Subgroups	Cosets	Right Cosets
$H_0 = S_3$	S_3	S_3
$H_1 = \{1, (132), (123)\}$	$H_1 = \{1, (132), (123)\}$ $(12)H_1 = \{(12), (13), (23)\}$	$H_1 = \{1, (132), (123)\}$ $H_1(12) = \{(12), (13), (23)\}$
$H_2 = \{1, (12)\}$	$H_2 = \{1, (12)\}$ $(23)H_2 = \{(23), (132)\}$ $(13)H_2 = \{(13), (123)\}$	$H_2 = \{1, (12)\}$ $H_2(23) = \{(23), (123)\}$ $H_2(13) = \{(13), (132)\}$
$H_3 = \{1, (13)\}$	$H_3 = \{1, (13)\}$ $(23)H_3 = \{(23), (123)\}$ $(12)H_3 = \{(12), (132)\}$	$H_3 = \{1, (13)\}$ $H_3(23) = \{(23), (132)\}$ $H_3(12) = \{(12), (123)\}$
$H_4 = \{1, (23)\}$	$H_4 = \{1, (23)\}$ $(12)H_4 = \{(12), (123)\}$ $(13)H_4 = \{(13), (132)\}$	$H_4 = \{1, (23)\}$ $H_4(12) = \{(12), (132)\}$ $H_4(13) = \{(13), (123)\}$
$H_5 = \{1\}$	$H_5 = \{1\}$ $(12)H_5 = \{(12)\}$ $(23)H_5 = \{(23)\}$ $(13)H_5 = \{(13)\}$ $(132)H_5 = \{(132)\}$ $(123)H_5 = \{(123)\}$	$H_5 = \{1\}$ $(12)H_5 = \{(12)\}$ $(23)H_5 = \{(23)\}$ $(13)H_5 = \{(13)\}$ $(132)H_5 = \{(132)\}$ $(123)H_5 = \{(123)\}$

Subgroup H_i	Normalizer N_{H_i}	Centralizer Z_{H_i}
$H_0 = S_3$	$H_0 = S_3$	$H_5 = \{1\}$
$H_1 = \{1, (132), (123)\}$	$H_0 = S_3$	$H_1 = \{1, (132), (123)\}$
$H_2 = \{1, (12)\}$	$H_2 = \{1, (12)\}$	$H_2 = \{1, (12)\}$
$H_3 = \{1, (13)\}$	$H_3 = \{1, (13)\}$	$H_3 = \{1, (13)\}$
$H_4 = \{1, (23)\}$	$H_4 = \{1, (23)\}$	$H_4 = \{1, (23)\}$
$H_5 = \{1\}$	$H_0 = S_3$	$H_0 = S_3$

Generators and relations

	Generators	Relations	Realization
D_3	x, y	$x^3 = y^2 = 1$ $(xy)^2 = 1$	$x = (123)$ $y = (12)$
S_3	s_1, s_2	$s_1^2 = s_2^2 = 1$ $s_1 s_2 s_1 = s_2 s_1 s_2$	$s_1 = y = (12)$ $s_2 = x^2 y = (23)$

Some Homomorphisms

Homomorphism	Kernel	Image
$\varphi_0\colon \begin{array}{rcl} S_3 & \to & (1) \\ s_1 & \mapsto & 1 \\ s_2 & \mapsto & 1 \end{array}$	$\ker\varphi_0 = S_3$	$\operatorname{im}\varphi_0 = (1)$
$\epsilon\colon \begin{array}{rcl} S_3 & \to & Z_2 \\ s_1 & \mapsto & -1 \\ s_2 & \mapsto & -1 \end{array}$	$\ker\epsilon = A_3$	$\operatorname{im}\epsilon = Z_2$
$\varphi_2\colon \begin{array}{rcl} S_3 & \to & O(3) \\ (12) & \mapsto & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ (23) & \mapsto & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$	$\ker\varphi_2 = (1)$	$\operatorname{im}\varphi_2 = \left\{ \begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \right\}$
$\varphi_3\colon \begin{array}{rcl} S_3 & \to & O(2) \\ (12) & \mapsto & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ (23) & \mapsto & \begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix} \end{array}$	$\ker\varphi_3 = (1)$	$\operatorname{im}\varphi_3 = \left\{ \begin{array}{cc} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix}, & \begin{pmatrix} -1/2 & -1/2 \\ 3/2 & -1/2 \end{pmatrix}, \\ \begin{pmatrix} -1/2 & 1/2 \\ -3/2 & -1/2 \end{pmatrix}, & \begin{pmatrix} -1/2 & -1/2 \\ -3/2 & -1/2 \end{pmatrix} \end{array} \right\}$
$\varphi_4\colon \begin{array}{rcl} S_3 & \to & D_3 \\ (12) & \mapsto & y \\ (132) & \mapsto & x \end{array}$	$\ker\varphi_4 = (1)$	$\operatorname{im}\varphi_4 = D_3$

The group action of D_3 as rotations and reflections of an equilateral triangle

D_3 is the group of rotations and reflections of an equilateral triangle. We shall denote the vertices by v_i, the edge connecting vertex i to vertex j by e_{ij}, $i < j$, and the face f_{012}. Let p_{ij}, $0 \leq i, j \leq 2$, denote the point on the edge connecting v_i to v_j which is a third of the way from v_i to v_j.

Let x be the 60° counterclockwise rotation about the center taking

$$v_0 \to v_1 \to v_2 \to v_0.$$

Let y be the reflection about the line connecting vertex v_0 with the midpoint of the edge e_{12}, taking

$$v_1 \to v_2 \quad \text{and fixing } v_0.$$

Note that $x^3 = 1$, $y^2 = 1$, and $yx = x^{-1}y$.
Let

$$\begin{aligned} P &= \{p_{01}, p_{10}, p_{12}, p_{21}, p_{02}, p_{20}\}, \\ V &= \{v_0, v_1, v_2\}, \\ E &= \{e_{01}, e_{12}, e_{02}\}, \qquad \text{and} \\ F &= \{f_{012}\}, \end{aligned}$$

denote the sets of points, vertices, edges, and faces, respectively. Since D_3 acts on the equilateral triangle, D_3 acts on each of these sets.

Stabilizer	Size of Stabilizer	Orbit	Size of Orbit
$(D_3)_{p_{ij}} = (1)$	1	$D_3 p_{ij} = P$	6
$(D_3)_{v_0} = \{1, y\} = H$	2	$D_3 v_0 = V$	3
$(D_3)_{v_1} = \{1, x^2 y\} = xHx^{-1}$	2	$D_3 v_1 = V$	3
$(D_3)_{v_2} = \{1, xy\} = x^2 H x^{-2}$	2	$D_3 v_2 = V$	3
$(D_3)_{e_{01}} = \{1, xy\} = x^2 H x^{-2}$	2	$D_3 e_{01} = E$	3
$(D_3)_{e_{12}} = \{1, y\} = H$	2	$D_3 e_{12} = E$	3
$(D_3)_{e_{02}} = \{1, x^2 y\} = xHx^{-1}$	2	$D_3 e_{02} = E$	3
$(D_3)_{f_{012}} = D_3$	6	$D_3 f_{012} = F$	1

§4T. The dihedral group D_4 of order 8

The group D_4 is as in the following table.

Set	Operation
$D_4 = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$	$x^i y^j x^k y^l = x^{(i-k) \bmod 4} y^{(j+l) \bmod 2}$

The complete multiplication tables for D_4 is as follows.

Multiplication Table

	1	x	x^2	x^3	y	xy	x^2y	x^3y
1	1	x	x^2	x^3	y	xy	x^2y	x^3y
x	x	x^2	x^3	1	xy	x^2y	x^3y	y
x^2	x^2	x^3	1	x	x^2y	x^3y	y	xy
x^3	x^3	1	x	x^2	x^3y	y	xy	x^2y
y	y	x^3y	x^2y	xy	1	x^3	x^2	x
xy	xy	y	x^3y	x^2y	x	1	x^3	x^2
x^2y	x^2y	xy	y	x^3y	x^2	x	1	x^3
x^3y	x^3y	x^2y	xy	y	x^3	x^2	x	1

Center	Abelian	Conjugacy Classes	Subgroups
$Z(D_4) = \{1, x^2\}$	No	$\mathcal{C}_1 = \{1\}$	$H_0 = D_4$
		$\mathcal{C}_{x^2} = \{x^2\}$	$H_1 = \{1, x, x^2, x^3\}$
		$\mathcal{C}_y = \{y, x^2y\}$	$H_2 = \{1, x^2, y, x^2y\}$
		$\mathcal{C}_{xy} = \{xy, x^3y\}$	$H_3 = \{1, x^2, xy, x^3y\}$
		$\mathcal{C}_x = \{x, x^3\}$	$H_4 = \{1, x^2\}$
			$H_5 = \{1, y\}$
			$H_6 = \{1, xy\}$
			$H_7 = \{1, x^2y\}$
			$H_8 = \{1, x^3y\}$
			$H_9 = \{1\}$

Elements

Element g	Order $o(g)$	Centralizer Z_g	Conjugacy Class $\mathcal{C}_g$
1	1	D_4	$\mathcal{C}_1$
x	4	H_1	$\mathcal{C}_x$
x^2	2	D_4	$\mathcal{C}_{x^2}$
x^3	4	H_1	$\mathcal{C}_x$
y	2	H_2	$\mathcal{C}_y$
xy	2	H_3	$\mathcal{C}_{xy}$
x^2y	2	H_2	$\mathcal{C}_y$
x^3y	2	H_3	$\mathcal{C}_{xy}$

Subgroup Lattice

Orders		**Inclusions**			
		D_4			
4	$\langle x \rangle = \{1, x, x^2, x^3\}$		$\{1, x^2, y, x^2y\}$	$\{1, x^2, xy, x^3y\}$	
2	$\langle x^2 \rangle = \{1, x^2\}$	$\langle y \rangle = \{1, y\}$	$\langle x^2y \rangle = \{1, x^2y\}$	$\langle xy \rangle = \{1, xy\}$	$\langle x^3y \rangle = \{1, x^3y\}$
1		$\{1\}$			

Subgroups	Structure	Index	Normal	Quotient Group
$H_0 = D_4$	$H_0 = D_4$	$[D_4 : D_4] = 1$	Yes	$D_4/H_0 \simeq (1)$
$H_1 = \{1, x, x^2, x^3\}$	$H_1 \simeq Z_4$	$[D_4 : H_1] = 2$	Yes	$D_4/H_0 \simeq Z_2$
$H_2 = \{1, x^2, y, x^2y\}$	$H_2 \simeq Z_2 \times Z_2$	$[D_4 : H_2] = 2$	Yes	$D_4/H_2 \simeq Z_2$
$H_3 = \{1, x^2, y, x^2y\}$	$H_3 \simeq Z_2 \times Z_2$	$[D_4 : H_3] = 2$	Yes	$D_4/H_3 \simeq Z_2$
$H_4 = \{1, x^2\}$	$H_4 \simeq Z_2$	$[D_4 : H_4] = 4$	Yes	$D_4/H_4 \simeq Z_2 \times Z_2$
$H_5 = \{1, y\}$	$H_5 \simeq Z_2$	$[D_4 : H_5] = 4$	No	
$H_6 = \{1, xy\}$	$H_6 \simeq Z_2$	$[D_4 : H_6] = 4$	No	
$H_7 = \{1, x^2y\}$	$H_7 \simeq Z_2$	$[D_4 : H_7] = 4$	No	
$H_8 = \{1, x^3y\}$	$H_8 \simeq Z_2$	$[D_4 : H_8] = 4$	No	
$H_9 = \{1\}$	$H_9 = \{1\}$	$[D_4 : \{1\}] = 8$	Yes	$D_4/(1) \simeq D_4$

Subgroups	Cosets	Right Cosets
$H_0 = D_4$	$D_4 = xD_4 = x^3D_4 = yD_4$ $= xyD_4 = x^2yD_4 = x^3yD_4$	$D_4 = D_4x = D_4x^2 = D_4x^3$ $= D_4y = D_4xy = D_4x^2y = D_4x^3y$
$H_1 = \{1, x, x^2, x^3\}$	$H_1 = xH_1 = x^2H_1 = x^3H_1$ $= \{1, x, x^2, x^3\}$ $yH_1 = xyH_1 = x^2yH_1 = x^3yH_1$ $= \{y, xy, x^2y, x^3y\}$	$H_1 = H_1x = H_1x^2 = H_1x^3$ $= \{1, x, x^2, x^3\}$ $H_1y = H_1xy = H_1x^2y = H_1x^3y$ $= \{y, xy, x^2y, x^3y\}$
$H_2 = \{1, x^2, y, x^2y\}$	$H_2 = x^2H_2 = yH_2 = x^2yH_2$ $= \{1, x^2, y, x^2y\}$ $xH_2 = x^3H_2 = xyH_2 = x^3yH_2$ $= \{x, x^3, xy, x^3y\}$	$H_2 = H_2x^2 = H_2y = H_2x^2y$ $= \{1, x^2, y, x^2y\}$ $H_2x = H_2x^3 = H_2xy = H_2x^3y$ $= \{x, x^3, xy, x^3y\}$
$H_3 = \{1, x^2, xy, x^3y\}$	$H_3 = x^2H_3 = xyH_3 = x^3yH_3$ $= \{1, x^2, xy, x^3y\}$ $xH_3 = x^3H_3 = yH_3 = x^2yH_3$ $= \{x, x^3, y, x^2y\}$	$H_3 = H_3x^2 = H_3xy = H_3x^3y$ $= \{1, x^2, xy, x^3y\}$ $H_3x = H_3x^3 = H_3y = H_3x^2y$ $= \{x, x^3, y, x^2y\}$
$H_4 = \{1, x^2\}$	$H_4 = x^2H_4 = \{1, x^2\}$ $xH_4 = x^3H_4 = \{x, x^3\}$ $yH_4 = x^2yH_4 = \{y, x^2y\}$ $xyH_4 = x^3yH_4 = \{xy, x^3y\}$	$H_4 = H_4x^2 = \{1, x^2\}$ $H_4x = H_4x^3 = \{x, x^3\}$ $H_4y = H_4x^2y = \{y, x^2y\}$ $H_4xy = H_4x^3y = \{xy, x^3y\}$
$H_5 = \{1, y\}$	$H_5 = yH_5 = \{1, y\}$ $xH_5 = xyH_5 = \{x, xy\}$ $x^2H_5 = x^2yH_5 = \{x^2, x^2y\}$ $x^3H_5 = x^3yH_5 = \{x^3, x^3y\}$	$H_5 = H_5y = \{1, y\}$ $H_5x = H_5x^3y = \{x, x^3y\}$ $H_5x^2 = H_5x^2y = \{x^2, x^2y\}$ $H_5x^3 = H_5xy = \{x^3, xy\}$
$H_6 = \{1, xy\}$	$H_6 = xyH_6 = \{1, xy\}$ $xH_6 = x^2yH_6 = \{x, x^2y\}$ $x^2H_6 = x^3yH_6 = \{x^2, x^3y\}$ $x^3H_6 = yH_6 = \{x^3, y\}$	$H_6 = H_6xy = \{1, xy\}$ $H_6x = H_6x^2y = \{x, y\}$ $H_6x^2 = H_6x^3y = \{x^2, x^3y\}$ $H_6x^3 = H_6x^2y = \{x^3, x^2y\}$
$H_7 = \{1, x^2y\}$	$H_7 = x^2yH_7 = \{1, x^2y\}$ $xH_7 = x^3yH_7 = \{x, x^3y\}$ $x^2H_7 = yH_7 = \{x^2, y\}$ $x^3H_7 = xyH_7 = \{x^3, xy\}$	$H_7 = H_7x^2y = \{1, x^2y\}$ $H_7x = H_7xy = \{x, xy\}$ $H_7x^2 = H_7y = \{x^2, y\}$ $H_7x^3 = H_7x^3y = \{x^3, x^3y\}$
$H_8 = \{1, x^3y\}$	$H_8 = x^3yH_8 = \{1, x^3y\}$ $xH_8 = yH_8 = \{x, y\}$ $x^2H_8 = xyH_8 = \{x^2, xy\}$ $x^3H_8 = x^2yH_8 = \{x^3, x^2y\}$	$H_8 = H_8x^3y = \{1, x^3y\}$ $H_8x = H_8x^2y = \{x, x^2y\}$ $H_8x^2 = H_8xy = \{x^2, xy\}$ $H_8x^3 = H_8y = \{x^3, y\}$
$H_9 = \{1\}$	$H_9 = \{1\}, xH_9 = \{x\},$ $x^2H_9 = \{x^2\}, x^3H_9 = \{x^3\}$ $yH_9 = \{y\}, xyH_9 = \{xy\},$ $x^2yH_9 = \{x^2y\}, x^3yH_9 = \{x^3y\}$	$H_9 = \{1\}, H_9x = \{x\},$ $H_9x^2 = \{x^2\}, H_9x^3 = \{x^3\}$ $H_9y = \{y\}, H_9xy = \{xy\},$ $H_9x^2y = \{x^2y\}, H_9x^3y = \{x^3y\}$

Subgroup H_i	Normalizer N_{H_i}	Centralizer Z_{H_i}
$H_0 = D_4$	D_4	$Z(D_4) = H_4 = \langle x^2 \rangle$
$H_1 = \langle x \rangle$	D_4	$H_1 = \langle x \rangle$
$H_2 = \langle x^2, y \rangle$	D_4	$H_2 = \langle x^2, y \rangle$
$H_3 = \langle x^2, xy \rangle$	D_4	$H_3 = \langle x^2, xy \rangle$
$H_4 = \langle x^2 \rangle$	D_4	D_4
$H_5 = \langle y \rangle$	$H_2 = \langle x^2, y \rangle$	$H_2 = \langle x^2, y \rangle$
$H_6 = \langle xy \rangle$	$H_3 = \langle x^2, xy \rangle$	$H_3 = \langle x^2, xy \rangle$
$H_7 = \langle x^2 y \rangle$	$H_2 = \langle x^2, y \rangle$	$H_2 = \langle x^2, y \rangle$
$H_8 = \langle x^3 y \rangle$	$H_3 = \langle x^2, xy \rangle$	$H_3 = \langle x^2, xy \rangle$
$H_9 = (1)$	D_4	D_4

Some Homomorphisms

Homomorphism	Kernel	Image
$\varphi_0\colon \begin{array}{ccc} D_4 & \to & (1) \\ x & \mapsto & 1 \\ y & \mapsto & 1 \end{array}$	$\ker \varphi_0 = D_4$	$\operatorname{im} \varphi_0 = (1)$
$\varphi_1\colon \begin{array}{ccc} D_4 & \to & Z_2 \\ x & \mapsto & 1 \\ y & \mapsto & -1 \end{array}$	$\ker \varphi_1 = H_1$	$\operatorname{im} \varphi_1 = Z_2$
$\varphi_2\colon \begin{array}{ccc} D_4 & \to & Z_2 \\ x & \mapsto & -1 \\ y & \mapsto & 1 \end{array}$	$\ker \varphi_2 = \{1, x^2, y, x^2 y\} = H_2$	$\operatorname{im} \varphi_2 = Z_2$
$\varphi_3\colon \begin{array}{ccc} D_4 & \to & Z_2 \\ x & \mapsto & -1 \\ y & \mapsto & -1 \end{array}$	$\ker \varphi_3 = \{1, x^2, xy, x^3 y\} = H_3$	$\operatorname{im} \varphi_3 = Z_2$
$\varphi_4\colon \begin{array}{ccc} D_4 & \to & Z_2 \times Z_2 \\ x & \mapsto & \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \\ y & \mapsto & \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \end{array}$	$\ker \varphi_4 = \{1, x^2\} = H_4$	$\operatorname{im} \varphi_4 = Z_2 \times Z_2$
$\varphi_9\colon \begin{array}{ccc} D_4 & \to & D_4 \\ x & \mapsto & x \\ y & \mapsto & y \end{array}$	$\ker \varphi_9 = \{1\} = H_9$	$\operatorname{im} \varphi_9 = D_4$

Generators and relations

Generators	Relations
x, y	$x^4 = y^2 = 1$ $yx = x^{-1} y$

The group action of D_4 as rotations and reflections of a square

D_4 is the group of rotations and reflections of the square. We shall denote the vertices by v_i, the edge connecting vertex i to vertex j by e_{ij}, $i < j$, and the face f_{0123}. For all v_i and v_j connected by an edge, let p_{ij} denote the point on the edge connecting v_i to v_j which is a third of the way from v_i to v_j.

Let x be the 90° counterclockwise rotation about the center taking

$$v_0 \to v_1 \to v_2 \to v_3 \to v_0.$$

Let y be the reflection about the line connecting vertex v_0 with vertex v_2, taking

$$v_1 \to v_3 \quad \text{and fixing } v_0 \text{ and } v_2.$$

Note that $x^4 = 1$, $y^2 = 1$, and $yx = x^{-1}y$.
Let

$$\begin{aligned} P &= \{p_{01}, p_{10}, p_{12}, p_{21}, p_{23}, p_{32}, p_{03}, p_{30}\}, \\ V &= \{v_0, v_1, v_2, v_3\}, \\ E &= \{e_{01}, e_{12}, e_{23}, e_{03}\}, \qquad \text{and} \\ F &= \{f_{0123}\}, \end{aligned}$$

denote the sets of points, vertices, edges, and faces, respectively. Since D_4 acts on the square, D_4 acts on each of these sets.

Stabilizer	Size of Stabilizer	Orbit	Size of Orbit
$(D_4)_{p_{ij}} = (1)$	1	$D_4 p_{ij} = P$	8
$(D_4)_{v_0} = \{1, y\} = H$	2	$D_4 v_0 = V$	4
$(D_4)_{v_1} = \{1, x^2 y\} = xHx^{-1}$	2	$D_4 v_1 = V$	4
$(D_4)_{v_2} = \{1, y\} = H$	2	$D_4 v_2 = V$	4
$(D_4)_{v_3} = \{1, x^2 y\} = xHx^{-1}$	2	$D_4 v_3 = V$	4
$(D_4)_{e_{01}} = \{1, xy\} = J$	2	$D_4 e_{01} = E$	4
$(D_4)_{e_{23}} = \{1, xy\} = J$	2	$D_4 e_{23} = E$	4
$(D_4)_{e_{12}} = \{1, x^3 y\} = xJx^{-1}$	2	$D_4 e_{12} = E$	4
$(D_4)_{e_{03}} = \{1, x^3 y\} = xJx^{-1}$	2	$D_4 e_{03} = E$	4
$(D_4)_{f_{0123}} = D_4$	8	$D_4 f_{0123} = F$	1

§5T. The quaternion group Q

The quaternion group Q is as in the following table. The element -1 acts like -1 in the complex numbers, it takes everything to its negative, and the negative of a negative is a positive.

Set	Operation
$Q = \{1, -1, i, -i, j, -j, k, -k\}$	$i^2 = j^2 = k^2 = ijk = -1$

The complete multiplication table for Q is as follows.

Multiplication Table

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

Center	Abelian	Conjugacy Classes	Subgroups
$Z(Q) = \{1, -1\}$	No	$\mathcal{C}_1 = \{1\}$	$H_0 = Q$
		$\mathcal{C}_{-1} = \{-1\}$	$H_1 = \{\pm 1, \pm i\}$
		$\mathcal{C}_i = \{\pm i\}$	$H_2 = \{\pm 1, \pm j\}$
		$\mathcal{C}_j = \{\pm j\}$	$H_3 = \{\pm 1, \pm k\}$
		$\mathcal{C}_k = \{\pm k\}$	$H_4 = \{\pm 1\}$
			$H_5 = \{1\}$

Elements

Element g	Order $o(g)$	Centralizer Z_g	Conjugacy Class $\mathcal{C}_g$
1	1	Q	$\mathcal{C}_1$
-1	2	Q	$\mathcal{C}_{-1}$
i	4	H_1	$\mathcal{C}_i$
$-i$	4	H_1	$\mathcal{C}_i$
j	4	H_2	$\mathcal{C}_j$
$-j$	4	H_2	$\mathcal{C}_j$
k	4	H_3	$\mathcal{C}_k$
$-k$	4	H_3	$\mathcal{C}_k$

Subgroups

Subgroups	Structure	Index	Normal	Quotient Group
$H_0 = Q$	$H_0 = Q$	$[Q : Q] = 1$	Yes	$Q/H_0 \simeq (1)$
$H_1 = \{\pm 1, \pm i\}$	$H_1 \simeq Z_4$	$[Q : H_1] = 2$	Yes	$Q/H_1 \simeq Z_2$
$H_2 = \{\pm 1, \pm j\}$	$H_2 \simeq Z_4$	$[Q : H_2] = 2$	Yes	$Q/H_2 \simeq Z_2$
$H_3 = \{\pm 1, \pm k\}$	$H_3 \simeq Z_4$	$[Q : H_3] = 2$	Yes	$Q/H_3 \simeq Z_2$
$H_4 = \{\pm 1\}$	$H_4 \simeq Z_2$	$[Q : H_4] = 4$	Yes	$Q/H_4 \simeq Z_2 \times Z_2$
$H_5 = \{1\}$	$H_5 = \{1\}$	$[Q : H_5] = 8$	Yes	$Q/(1) \simeq Q$

Subgroups	Cosets	Right Cosets
$H_0 = Q$	Q	Q
$H_1 = \{\pm 1, \pm i\}$	$H_1 = \{\pm 1, \pm i\}$ $jH_1 = \{\pm j, \pm k\}$	$H_1 = \{\pm 1, \pm i\}$ $H_1 j = \{\pm j, \pm k\}$
$H_2 = \{\pm 1, \pm j\}$	$H_2 = \{\pm 1, \pm j\}$ $iH_2 = \{\pm i, \pm k\}$	$H_2 = \{\pm 1, \pm j\}$ $H_2 i = \{\pm i, \pm k\}$
$H_3 = \{\pm 1, \pm k\}$	$H_3 = \{\pm 1, \pm k\}$ $iH_3 = \{\pm i, \pm j\}$	$H_3 = \{\pm 1, \pm k\}$ $H_3 i = \{\pm i, \pm j\}$
$H_4 = \{\pm 1\}$	$H_4 = \{\pm 1\}$ $iH_4 = \{\pm i\}$ $jH_4 = \{\pm j\}$ $kH_4 = \{\pm k\}$	$H_4 = \{\pm 1\}$ $H_4 i = \{\pm i\}$ $H_4 j = \{\pm j\}$ $H_4 k = \{\pm k\}$
$H_5 = \{1\}$	$H_5 = \{1\}$ $(-1)H_5 = \{-1\}$ $iH_5 = \{i\}$ $-iH_5 = \{-i\}$ $jH_5 = \{j\}$ $-jH_5 = \{-j\}$ $kH_5 = \{k\}$ $-kH_5 = \{-k\}$	$H_5 = \{1\}$ $H_5(-1) = \{-1\}$ $H_5 i = \{i\}$ $H_5(-i) = \{-i\}$ $H_5 j = \{j\}$ $H_5(-j) = \{-j\}$ $H_5 k = \{k\}$ $H_5(-k) = \{-k\}$

Subgroup H_i	Normalizer N_{H_i}	Centralizer Z_{H_i}
$H_0 = Q$	Q	$H_4 = \{\pm 1\}$
$H_1 = \langle i \rangle$	Q	$H_1 = \langle i \rangle$
$H_2 = \langle j \rangle$	Q	$H_2 = \langle j \rangle$
$H_3 = \langle k \rangle$	Q	$H_3 = \langle k \rangle$
$H_4 = \{\pm 1\}$	Q	Q
$H_5 = (1)$	Q	Q

Subgroup Lattice

Orders		Inclusions	
8		Q	
4	$\{\pm 1, \pm i\}$	$\{\pm 1, \pm j\}$	$\{\pm 1, \pm k\}$
2		$\{\pm 1\}$	
1		$\{1\}$	

Generators and relations

Generators	Relations	Realization
S, T	$S^2 = T^2 = (ST)^2$	$S = i, T = j, ST = k$

Some Homomorphisms

Homomorphism	Kernel	Image
$\varphi_0\colon \begin{array}{ccc} Q & \to & (1) \\ i & \mapsto & 1 \\ j & \mapsto & 1 \end{array}$	$\ker \varphi_0 = Q$	$\operatorname{im} \varphi_0 = (1)$
$\varphi_1\colon \begin{array}{ccc} Q & \to & Z_2 \\ i & \mapsto & 1 \\ j & \mapsto & -1 \end{array}$	$\ker \varphi_1 = H_1 = \{\pm 1, \pm i\}$	$\operatorname{im} \varphi_1 = Z_2$
$\varphi_2\colon \begin{array}{ccc} Q & \to & Z_2 \\ i & \mapsto & -1 \\ j & \mapsto & 1 \end{array}$	$\ker \varphi_2 = H_2 = \{\pm 1, \pm j\}$	$\operatorname{im} \varphi_2 = Z_2$
$\varphi_3\colon \begin{array}{ccc} Q & \to & Z_2 \\ i & \mapsto & -1 \\ j & \mapsto & -1 \end{array}$	$\ker \varphi_3 = H_3 = \{\pm 1, \pm k\}$	$\operatorname{im} \varphi_3 = Z_2$
$\varphi_4\colon \begin{array}{ccc} Q & \to & Gl_2(\mathbb{C}) \\ i & \mapsto & \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ j & \mapsto & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ k & \mapsto & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \end{array}$	$\ker \varphi_4 = H_5 = (1)$	$\operatorname{im} \varphi_4 = \left\{ \begin{array}{c} \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix}, \\ \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \pm i \\ \pm i & 0 \end{pmatrix} \end{array} \right\}$
$\varphi_5\colon \begin{array}{ccc} Q & \to & Z_2 \times Z_2 \\ i & \mapsto & (-1, 1) \\ j & \mapsto & (1, -1) \end{array}$	$\ker \varphi_5 = H_4 = \{\pm 1\}$	$\operatorname{im} \varphi_5 = Z_2 \times Z_2$

The group A_4 can be given in at least two natural ways. In the following tables we shall use one-line notation to represent the permutations in A_4.

Set	Operation
even permutations in S_4	composition of permutations
rotations preserving a tetrahedron	composition of rotations

Center	Abelian	Conjugacy Classes	Subgroups
$Z(A_4) = \{(1234)\}$	No	$\mathcal{C}_{(1^4)} = \{(1234)\}$	$H_0 = A_4$
		$\mathcal{C}_{(2^2)} = \{(2143), (3412), (4321)\}$	
		$\mathcal{C}_{(31)^+} = \{(3124), (4213), (2431), (1342)\}$	
		$\mathcal{C}_{(31)^-} = \{(2314), (3241), (4132), (1423)\}$	

Subgroups
$H_1 = \{(1234), (2143), (3412), (4321)\}$
$H_2 = \{(1234), (3124), (2314)\}$
$H_3 = \{(1234), (4132), (2431)\}$
$H_4 = \{(1234), (4213), (3241)\}$
$H_5 = \{(1234), (1423), (1342)\}$
$H_6 = \{(1234), (3412)\}$
$H_7 = \{(1234), (2143)\}$
$H_8 = \{(1234), (4321)\}$
$H_9 = \{(1234)\}$

Elements

Element g	Order $o(g)$	Centralizer Z_g	Conjugacy Class $\mathcal{C}_g$
(1234)	1	A_4	$\mathcal{C}_{(1^4)}$
(2143)	2	H_1	$\mathcal{C}_{2^2}$
(3412)	2	H_1	$\mathcal{C}_{2^2}$
(4321)	2	H_1	$\mathcal{C}_{2^2}$
(3124)	3	H_2	$\mathcal{C}_{(31)^+}$
(4213)	3	H_4	$\mathcal{C}_{(31)^+}$
(2431)	3	H_3	$\mathcal{C}_{(31)^+}$
(1342)	3	H_5	$\mathcal{C}_{(31)^+}$
(2314)	3	H_2	$\mathcal{C}_{(31)^-}$
(3241)	3	H_4	$\mathcal{C}_{(31)^-}$
(4132)	3	H_3	$\mathcal{C}_{(31)^-}$
(1423)	3	H_5	$\mathcal{C}_{(31)^-}$

Subgroups

Subgroups	Structure	Index	Normal	Quotient Group
$H_0 = A_4$	$H_0 = A_4$	$[A_4 : A_4] = 1$	Yes	$A_4/A_4 \simeq (1)$
$H_1 = \{(1234), (2143), (3412), (4321)\}$	$H_1 \simeq Z_2 \times Z_2$	$[A_4 : H_1] = 3$	Yes	$A_4/H_1 \simeq Z_3$
$H_2 = \{(1234), (3124), (2314)\}$	$H_2 \simeq Z_3$	$[A_4 : H_2] = 4$	No	
$H_3 = \{(1234), (4132), (2431)\}$	$H_3 \simeq Z_3$	$[A_4 : H_3] = 4$	No	
$H_4 = \{(1234), (4213), (3241)\}$	$H_4 \simeq Z_3$	$[A_4 : H_4] = 4$	No	
$H_5 = \{(1234), (1423), (1342)\}$	$H_5 \simeq Z_3$	$[A_4 : H_5] = 4$	No	
$H_6 = \{(1234), (3412)\}$	$H_6 \simeq Z_2$	$[A_4 : H_6] = 6$	No	
$H_7 = \{(1234), (2143)\}$	$H_7 \simeq Z_2$	$[A_4 : H_7] = 6$	No	
$H_8 = \{(1234), (4321)\}$	$H_8 \simeq Z_2$	$[A_4 : H_8] = 6$	No	
$H_9 = \{(1234)\}$	$H_9 \simeq (1)$	$[A_4 : H_9] = 12$	Yes	$A_4/(1) \simeq A_4$

Subgroup Lattice

Orders	**Inclusions**
12	A_4
4	H_1
3	H_2 H_3 H_4 H_5
2	H_6 H_7 H_8
1	H_9

Subgroup H_i	Normalizer N_{H_i}	Centralizer Z_{H_i}
$H_0 = A_4$	A_4	$H_9 = (1)$
N	A_4	N
H_2	A_4	H_2
H_3	A_4	H_3
H_4	A_4	H_4
H_5	A_4	H_5
H_6	N	H_1
H_7	N	H_1
H_8	N	H_1
$H_9 = (1)$	A_4	A_4

Generators and relations

Generators	Relations	Realization
S, T	$S^3 = T^2 = (ST)^3 = 1$	$S = (2314), T = (2143)$

Some Homomorphisms

Let w be the primitive cube root of 1 given by $w = e^{2\pi i/3} \in \mathbb{C}$.

Homomorphism	Kernel
$\varphi_0\colon \begin{array}{ccc} A_4 & \to & (1) \\ S & \mapsto & 1 \\ T & \mapsto & 1 \end{array}$	$\ker \varphi_0 = A_4$
$\varphi_1\colon \begin{array}{ccc} A_4 & \to & Z_3 \\ S & \mapsto & w \\ T & \mapsto & 1 \end{array}$	$\ker \varphi_1 = H_1$
$\varphi_2\colon \begin{array}{ccc} A_4 & \to & Z_3 \\ S & \mapsto & w^2 \\ T & \mapsto & 1 \end{array}$	$\ker \varphi_2 = H_1$
$\varphi_3\colon \begin{array}{ccc} A_4 & \to & GL(3) \\ S & \mapsto & \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & -3/2 \\ 0 & 1/2 & -1/2 \end{pmatrix} \\ T & \mapsto & \begin{pmatrix} -1/3 & -4/3 & 0 \\ -2/3 & 1/3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{array}$	$\ker \varphi_3 = (1)$

The group action of A_4 as rotations of a tetrahedron

A_4 is the group of rotations of the tetrahedron. We shall denote the vertices by v_i, the edge connecting vertex i to vertex j by e_{ij}, $i < j$, and the face adjacent to the three vertices v_i, v_j, v_k, by f_{ijk}, $i < j < k$. Let r_{1234} denote the region determined by the inside of the tetrahedron. Let p_{ij}, $1 \leq i, j \leq 4$ denote the point on the edge connecting v_i to v_j which is a third of the way from v_i to v_j.

Let S be the 60° rotation about the bottom face taking

$$v_1 \to v_2 \to v_3 \to v_1 \quad \text{and fixing } v_4.$$

Let T be the 180° rotation about the line connecting the midpoint of edge e_{34} with the midpoint of edge e_{12}, taking

$$v_1 \to v_2 \quad \text{and} \quad v_3 \to v_4.$$

Note that $S^3 = 1$, $T^2 = 1$, and $(ST)^3 = 1$.
Let

$$\begin{aligned} P &= \{p_{ij} \mid 1 \leq i, j \leq 4\}, \\ V &= \{v_1, v_2, v_3, v_4\}, \\ E &= \{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}, \\ F &= \{f_{123}, f_{124}, f_{134}, f_{234}\}, \qquad \text{and} \\ R &= \{r_{1234}\}, \end{aligned}$$

denote the sets of points, vertices, edges, faces, and regions, respectively. Since A_4 acts on the tetrahedron, A_4 acts on each of these sets.

Stabilizer	Size of Stabilizer	Orbit	Size of Orbit
$(A_4)_{p_{ij}} = (1)$	1	$A_4 p_{ij} = P$	12
$(A_4)_{v_4} = \{1, S, S^2\} = H$	3	$A_4 v_4 = V$	4
$(A_4)_{v_3} = \{1, TST^{-1}, TS^2T^{-1}\} = THT^{-1}$	3	$A_4 v_3 = V$	4
$(A_4)_{v_1} = \{1, TS, S^2T\} = (ST)H(ST)^{-1}$	3	$A_4 v_1 = V$	4
$(A_4)_{v_2} = \{1, ST, (ST)^2\} = (S^2T)H(S^2T)^{-1}$	3	$A_4 v_2 = V$	4
$(A_4)_{e_{12}} = \{1, T\}$	2	$A_4 e_{12} = E$	6
$(A_4)_{e_{34}} = \{1, T\}$	2	$A_4 e_{34} = E$	6
$(A_4)_{e_{14}} = \{1, STS^{-1}\}$	2	$A_4 e_{14} = E$	6
$(A_4)_{e_{23}} = \{1, STS^{-1}\}$	2	$A_4 e_{23} = E$	6
$(A_4)_{e_{13}} = \{1, S^2TS^{-2}\}$	2	$A_4 e_{13} = E$	6
$(A_4)_{e_{24}} = \{1, S^2TS^{-2}\}$	2	$A_4 e_{24} = E$	6
$(A_4)_{f_{123}} = \{1, S, S^2\}$	3	$A_4 f_{123} = F$	4
$(A_4)_{f_{124}} = \{1, TST^{-1}, TS^2T^{-1}\}$	3	$A_4 f_{124} = F$	4
$(A_4)_{f_{234}} = \{1, (ST)S(ST)^{-1}, (ST)S^2(ST)^{-1}\}$	3	$A_4 f_{234} = F$	4
$(A_4)_{f_{134}} = \{1, (S^2T)S(S^2T)^{-1}, (S^2T)S^2(S^2T)^{-1}\}$	3	$A_4 f_{134} = F$	4
$(A_4)_{r_{1234}} = A_4$	12	$A_4 r_{1234} = R$	1

§7T. The octahedral group S_4

The group S_4 can be represented in several different ways. Some of these are given in the following table.

Set	Operation
permutations of 4 elements	composition of permutations
rotations preserving a cube	composition of rotations
rotations preserving an octahedron	composition of rotations

The complete multiplication table for S_4 is a 24×24 matrix. This matrix is too big to include here. In the following tables we shall use one-line notation to represent the permutations in S_4.

Center	Abelian	Conjugacy classes
$Z(S_4) = \{1, -1\}$	No	$\mathcal{C}_{(1^4)} = \{(1234)\}$ $\mathcal{C}_{(21^2)} = \{(2134), (3214), (4231)(1324), (1432), (1243)\}$ $\mathcal{C}_{(2^2)} = \{(2143), (3412), (4321)\}$ $\mathcal{C}_{(31)} = \{(3124), (4132), (4213), (1423)(2314), (2431), (3241), (1342)\}$ $\mathcal{C}_{(4)} = \{(4123), (3142), (2413), (4312), (2341), (3421)\}$

Subgroups

There are more than 30 subgroups of the group S_4. We shall not give a list of all of the subgroups and we shall not give a subgroup lattice here. The following table lists only the normal subgroups of S_4.

Subgroups	Structure	Index	Normal	Quotient Group
$N_0 = S_4$	$N_0 = S_4$	$[S_4 : S_4] = 1$	Yes	$S_4/S_4 \simeq (1)$
$N_1 = A_4$	$N_1 = A_4$	$[S_4 : A_4] = 2$	Yes	$S_4/A_4 \simeq Z_2$
$N_2 = \{(1234), (2143), (3412), (4321)\}$	$N_2 \simeq Z_2 \times Z_2$	$[S_4 : N_2] = 6$	Yes	$S_4/N_2 \simeq S_3$
$N_3 = \{(1234)\}$	$N_3 \simeq (1)$	$[S_4 : H_9] = 24$	Yes	$S_4/(1) \simeq S_4$

Generators and relations

The following table gives two useful presentations of the octahedral group S_4.

Generators	Relations	Realization
S, T	$S^4 = T^2 = (ST)^3 = 1$	$S = (4123), T = 4231$
s_1, s_2, s_3	$s_1^2 = s_2^2 = s_3^2 = 1$ $s_1 s_2 s_1 = s_2 s_1 s_2$ $s_2 s_3 s_2 = s_3 s_2 s_3$	$s_1 = (2134)$ $s_2 = (1324)$ $s_3 = (1243)$

Some Homomorphisms

In the following table $s_1 = (2134)$, $s_2 = (1324)$, $s_3 = (1243)$ denote the simple transpositions in the group S_4. These simple transpositions generate S_4. Note also that the homomorphism labeled $\phi_{(1^4)}$ is the sign homomorphism ε of the symmetric group S_4.

Homomorphism	Kernel
$\varphi\colon S_4 \to S_3$; $s_1 \mapsto (213)$, $s_2 \mapsto (132)$, $s_3 \mapsto (213)$	$\ker\varphi = N_2$
$\varphi_{(4)}\colon S_4 \to (1)$; $s_1 \mapsto 1$, $s_2 \mapsto 1$, $s_3 \mapsto 1$	$\ker\varphi_{(4)} = S_4$
$\varphi_{(1^4)}\colon S_4 \to Z_2$; $s_1 \mapsto -1$, $s_2 \mapsto -1$, $s_3 \mapsto -1$	$\ker\varphi_{(1^4)} = A_4$
$\varphi_{(21^2)}\colon S_4 \to GL_3$; $s_1 \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $s_2 \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1/2 & 3/2 \\ 0 & 1/2 & -1/2 \end{pmatrix}$, $s_3 \mapsto \begin{pmatrix} 1/3 & 4/3 & 0 \\ 2/3 & -1/3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\ker\varphi_{(21^2)} = (1)$
$\varphi_{31}\colon S_4 \to GL_3$; $s_1 \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $s_2 \mapsto \begin{pmatrix} 1/2 & 3/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $s_3 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 4/3 \\ 0 & 2/3 & -1/3 \end{pmatrix}$	$\ker\varphi_{(31)} = (1)$
$\varphi_{(22)}\colon S_4 \to GL(2)$; $s_1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $s_2 \mapsto \begin{pmatrix} 1/2 & 3/2 \\ 1/2 & -1/2 \end{pmatrix}$, $s_3 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\ker\varphi_{(22)} = N_2$

The group action of S_4 as rotations of a cube

S_4 is the group of rotations of the cube. We shall denote the vertices by v_i, the edge connecting vertex i to vertex j by e_{ij}, $i < j$, and the face adjacent to the four vertices v_i, v_j, v_k, v_l, by f_{ijkl}, $i < j < k < l$. Let $r_{12345678}$ denote the region determined by the inside of the cube. For all v_i and v_j connected by an edge, let p_{ij}, denote the point on the edge connecting v_i to v_j which is a third of the way from v_i to v_j.

Let S be the 90° rotation about the top face taking

$$v_1 \to v_2 \to v_3 \to v_4 \to v_1 \quad \text{and}$$
$$v_5 \to v_6 \to v_7 \to v_8 \to v_5.$$

Let T be the rotation 90° about the right face taking

$$v_4 \to v_1 \to v_5 \to v_8 \quad \text{and}$$
$$v_3 \to v_2 \to v_6 \to v_7.$$

Let

$$\begin{aligned} P &= \{p_{ij} \mid 1 \leq i, j \leq 8\}, \\ V &= \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}, \\ E &= \{e_{12}, e_{23}, e_{34}, e_{14}, e_{15}, e_{48}, e_{26}, e_{37}, e_{56}, e_{67}, e_{78}, e_{58}\}, \\ F &= \{f_{1234}, f_{5678}, f_{1256}, f_{3478}, f_{1458}, f_{2367}\}, \qquad \text{and} \\ R &= \{r_{12345678}\}, \end{aligned}$$

denote the sets of points, vertices, edges, faces, and regions, respectively. Since S_4 acts on the cube, S_4 acts on each of these sets.

Stabilizer	Size of Stabilizer	Orbit	Size of Orbit
$(S_4)_{p_{ij}} = (1)$	1	$S_4 p_{ij} = P$	24
$(S_4)_{v_1} = \{1, T^3S, TS^3\} = H$	3	$S_4 v_1 = V$	8
$(S_4)_{v_7} = \{1, T^3S, TS^3\} = H$	3	$S_4 v_7 = V$	8
$(S_4)_{v_2} = \{1, S^3T^3, TS\} = SHS^{-1}$	3	$S_4 v_2 = V$	8
$(S_4)_{v_8} = \{1, S^3T^3, TS\} = SHS^{-1}$	3	$S_4 v_8 = V$	8
$(S_4)_{v_3} = \{1, ST, S^2TS\} = S^2HS^{-2}$	3	$S_4 v_3 = V$	8
$(S_4)_{v_5} = \{1, ST, S^2TS\} = S^2HS^{-2}$	3	$S_4 v_5 = V$	8
$(S_4)_{v_4} = \{1, S^3T, S^2TS^3\} = S^3HS^{-3}$	3	$S_4 v_4 = V$	8
$(S_4)_{v_6} = \{1, S^3T, S^2TS^3\} = S^3HS^{-3}$	3	$S_4 v_6 = V$	8
$(S_4)_{e_{12}} = \{1, TS^2\} = J$	2	$S_4 e_{12} = E$	12
$(S_4)_{e_{78}} = \{1, TS^2\} = J$	2	$S_4 e_{78} = E$	12
$(S_4)_{e_{23}} = \{1, STS\} = SJS^{-1}$	2	$S_4 e_{23} = E$	12
$(S_4)_{e_{58}} = \{1, STS\} = SJS^{-1}$	2	$S_4 e_{58} = E$	12
$(S_4)_{e_{34}} = \{1, S^2T\} = S^2JS^{-2}$	2	$S_4 e_{34} = E$	12
$(S_4)_{e_{56}} = \{1, S^2T\} = S^2JS^{-2}$	2	$S_4 e_{56} = E$	12
$(S_4)_{e_{14}} = \{1, S^3TS^3\} = S^3JS^{-3}$	2	$S_4 e_{14} = E$	12
$(S_4)_{e_{67}} = \{1, S^3TS^3\} = S^3JS^{-3}$	2	$S_4 e_{67} = E$	12
$(S_4)_{e_{15}} = \{1, ST^2\} = (ST^3)J(ST^3)^{-1}$	2	$S_4 e_{15} = E$	12
$(S_4)_{e_{37}} = \{1, ST^2\} = (ST^3)J(ST^3)^{-1}$	2	$S_4 e_{37} = E$	12
$(S_4)_{e_{48}} = \{1, S^3T^2\} = (S^3TS)J(S^3TS)^{-1}$	2	$S_4 e_{48} = E$	12
$(S_4)_{e_{26}} = \{1, S^3T^2\} = (S^3TS)J(S^3TS)^{-1}$	2	$S_4 e_{26} = E$	12
$(S_4)_{f_{1234}} = \{1, S, S^2, S^3\} = K$	4	$S_4 f_{1234} = F$	6
$(S_4)_{f_{5678}} = \{1, S, S^2, S^3\} = K$	4	$S_4 f_{5678} = F$	6
$(S_4)_{f_{1256}} = \{1, S^2T^2, S^3TS, STS^3\} = TKT^{-1}$	4	$S_4 f_{1256} = F$	6
$(S_4)_{f_{3478}} = \{1, S^2T^2, S^3TS, STS^3\} = TKT^{-1}$	4	$S_4 f_{3478} = F$	6
$(S_4)_{f_{1458}} = \{1, T, T^2, T^3\} = (ST^3)K(ST^3)^{-1}$	4	$S_4 f_{1458} = F$	6
$(S_4)_{f_{2367}} = \{1, T, T^2, T^3\} = (ST^3)K(ST^3)^{-1}$	4	$S_4 f_{2367} = F$	6
$(S_4)_{r_{12345678}} = S_4$	24	$S_4 r_{12345678} = R$	1