(1.3.1) Definition.

- Let $[1,m]$ denote the set $\{1,2,\ldots,m\}$. A **permutation** of m is a bijective map
$$\sigma\colon\ [1,m]\to[1,m].$$
- The **symmetric group**, S_m, is the set of permutations of m with the operation of composition of functions.

HW: Show that the order of the symmetric group S_m is $m! = m(m-1)(m-2)\cdots 2\cdot 1$.

There are several convenient ways of representing a permutation σ.

1) As a two line array $\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & m \\ \sigma(1) & \sigma(2) & \sigma(3) & \ldots & \sigma(m)\end{pmatrix}$.
2) As a one line array $\sigma = \big(\sigma(1)\sigma(2)\ldots\sigma(m)\big)$.
3) As an $m\times m$ matrix which has the $\big(\sigma(i),i\big)^{\text{th}}$ entry equal to 1 for all i and all other entries equal to 0.
4) As a function diagram consisting of two rows, of m dots each, such that the i^{th} dot of the upper row is connected by an edge to the $\sigma(i)^{\text{th}}$ dot of the lower row.
5) In cycle notation, as a collection of sequences $(i_1,i_2,\ldots,i_k)$ such that $\sigma(i_1)=i_2$, $\sigma(i_2)=i_3,\ldots,$ $\sigma(i_{k-1})=i_k$, $\sigma(i_k)=i_1$. We often leave out the cycles containing only one element when we write σ in cycle notation.

HW: Show that, in function diagram notation, the product $\tau\sigma$ of two permutations τ and σ is given by placing the diagram of σ above the diagram of τ and connecting the bottom dots of σ to the top dots of τ.

HW: Show that, in function diagram notation, the identity permutation is represented by m vertical lines.

HW: Show that, in function diagram notation, σ^{-1} is represented by the diagram of σ flipped over.

HW: Show that, in matrix notation, the product $\tau\sigma$ of two permutations τ and σ is given by matrix multiplication.

HW: Show that, in matrix notation, the identity permutation is the diagonal matrix with all 1's on the diagonal.

HW: Show that, in matrix notation, the matrix of σ^{-1} is the transpose of the matrix of σ.

HW: Show that the matrix of a permutation is always an orthogonal matrix.

Sign of a permutation

(1.3.2) Proposition. *For each permutation $\sigma\in S_m$, let $\det(\sigma)$ denote the determinant of the matrix which represents the permutation σ. The map*
$$\begin{array}{rccc}\varepsilon\colon & S_m & \to & \{\pm1\}\\ & \sigma & \mapsto & \det(\sigma)\end{array}$$
is a homomorphism from the symmetric group S_m to the group $\mathbb{Z}_2=\{\pm1\}$.

(1.3.3) Definition.

- The **sign homomorphism** of the symmetric group S_m is the homomorphism
$$\begin{array}{rccc}\varepsilon\colon & S_m & \to & \pm1\\ & \sigma & \mapsto & \det(\sigma)\end{array}$$
where $\det(\sigma)$ denote the determinant of the matrix which represents the permutation σ.
- The **sign** of a permutation σ is the determinant $\varepsilon(\sigma)$ of the permutation matrix representing σ.
- A permutation σ is **even** if $\varepsilon(\sigma)=+1$ and is **odd** if $\varepsilon(\sigma)=-1$.

Conjugacy Classes

(1.3.4) Definition.

- A **partition** $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of m is a weakly decreasing sequence of positive integers which sum to m, i.e.

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0, \qquad \text{and} \qquad \sum_{i=1}^{k} \lambda_i = m.$$

  The elements of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ are the **parts** of the partition λ. Sometimes we represent a partition λ in the form $\lambda = (1^{m_1} 2^{m_2} \cdots)$ if λ has m_1 1's, m_2 2's, and so on. We write $\lambda \vdash m$ if λ is a partition of m.

- The **cycles** of a permutation σ are the ordered sequences $(i_1, i_2, \ldots, i_k)$ such that $\sigma(i_1) = i_2, \sigma(i_2) = i_3, \ldots, \sigma(i_{k-1}) = i_k, \sigma(i_k) = i_1$.

- The **cycle type** $\tau(\sigma)$ of a permutation $\sigma \in S_m$ is the partition of m determined by the sizes of the cycles of σ.

Example. A permutation σ can have several different representations in cycle notation. In cycle notation,

$$(12345)(67)(89)(10), \qquad (51234)(67)(89), \qquad (45123)(67)(89)(10),$$
$$(34512)(89)(67), \quad \text{and} \quad (34512)(10)(98)(67),$$

all represent the same permutation in S_{10}, which, in two line notation, is given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 5 & 1 & 7 & 6 & 9 & 8 & 10 \end{pmatrix}.$$

Example. If σ is the permutation in S_9 which is given, in cycle notation, by

$$\sigma = (1362)(587)(49)$$

and π is the permutation in S_9 which is given, in 2-line notation, by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 6 & 1 & 3 & 5 & 9 & 2 & 8 & 7 \end{pmatrix},$$

then $\pi\sigma\pi^{-1}$ is the permutation which is given, in cycle notation, by

$$\pi\sigma\pi^{-1} = (4196)(582)(37) = (1964)(258)(37).$$

(1.3.5) Theorem.

a) *The conjugacy classes of S_m are the sets*

$$\mathcal{C}_\lambda = \{ \textit{ permutations } \sigma \textit{ with cycle type } \lambda \},$$

where λ is a partition of m.

b) *If $\lambda = (1^{m_1} 2^{m_2} \cdots)$ then the size of the conjugacy class $\mathcal{C}_\lambda$ is*

$$|\mathcal{C}_\lambda| = \frac{m!}{m_1! 1^{m_1} m_2! 2^{m_2} m_3! 3^{m_3} \cdots}.$$

The proof of Theorem (1.3.5) will use the following lemma.

(1.3.6) Lemma. *Suppose $\sigma \in S_m$ has cycle type $\lambda = (\lambda_1, \lambda_2, \ldots)$ and let γ_λ be the permutation in S_m which is given, in cycle notation, by*

$$\gamma_\lambda = (1, 2, \cdots, \lambda_1)(\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + 1, \cdots) \cdots.$$

a) Then σ is conjugate to γ_λ.
b) If $\tau \in S_m$ is conjugate to σ then τ has cycle type λ.
c) Suppose that $\lambda = (1^{m_1} 2^{m_2} \cdots)$. Then the order of the stabilizer of the permutation γ_λ, under the action of S_m on itself by conjugation, is

$$1^{m_1} m_1! 2^{m_2} m_2! \cdots.$$

Example. The sequence $\lambda = (66433322111)$ is a partition of 32 and can also be represented in the form $\lambda = (1^3 2^2 3^3 4\, 5^0 6^2) = (1^3 2^2 3^3 4\, 6^2)$. The conjugacy class $\mathcal{C}_\lambda$ in S_{32} has $\dfrac{32!}{1^3 \cdot 3! \cdot 2^2 \cdot 2! \cdot 3^3 \cdot 3! \cdot 4 \cdot 6^2 \cdot 2!}$ elements.

Generators and relations

(1.3.7) Definition.

- The **simple transpositions** in S_m are the elements $s_i = (i, i+1)$, $1 \le i \le m-1$.

(1.3.8) Proposition.

a) S_m is generated by the simple transpositions s_i, $1 \le i \le m-1$.
b) The simple transpositions s_i, $1 \le i \le m-1$, in S_m satisfy the relations

$$s_i s_j = s_j s_i, \qquad \text{if } |i-j| > 1,$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad 1 \le i \le m-2,$$
$$s_i^2 = 1, \qquad 1 \le i \le m-1.$$

(1.3.9) Definition.

- A **reduced word** for $\sigma \in S_m$ is an expression

$$\sigma = s_{i_1} \ldots s_{i_p}$$

  of σ as a product of simple transpositions such that the number of factors is as small as possible.
- The **length** $\ell(\sigma)$ of σ is the number of factors in a reduced word for the permutation σ.
- The set of **inversions** of σ is the set

$$\mathrm{inv}(\sigma) = \{(i,j) | 1 \le i < j \le m, \sigma(i) > \sigma(j)\}.$$

HW: Show that the sign $\varepsilon(s_i)$ of a simple transposition s_i in the symmetric group S_n is -1.

(1.3.10) Proposition. *Let σ be a permutation. Let $\ell(\sigma)$ be the length of σ and let* $\mathrm{inv}(\sigma)$ *be the set of inversions of the permutation σ. Then*

a) The sign of σ is $\varepsilon(\sigma) = (-1)^{\ell(\sigma)}$.
b) $Card(\mathrm{inv}(\sigma)) = \ell(\sigma)$
c) The number of crossings in the function diagram of σ is $\ell(\sigma)$.

(1.3.11) Theorem. *The symmetric group S_m has a presentation by generators, $s_1, s_2, \ldots, s_{m-1}$ and relations*

$$s_i s_j = s_j s_i, \qquad \text{if } |i-j| > 1.$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$
$$s_i^2 = 1.$$