MATH 221: Calculus and Analytic Geometry
Prof. Ram, Fall 2004

HOMEWORK 11
DUE November 22, 2004

Problem A. Volumes by washers.

(1) Show that the volume of a right circular cylinder of radius r and height h is $\pi r^2 h$ by using the washer method.

(2) Show that the volume of a right circular cone of radius r and height h is $\frac{1}{3}\pi r^2 h$ by using the washer method.

(3) Show that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ by using the washer method.

(4) Find the volume generated when the area bounded by the lines $x + y = 2$, $x = 0$, $y = 0$ is rotated about the x-axis.

(5) Find the volume generated when the area bounded by $y = \sin x$, $0 \le x \le \pi$, and $y = 0$ is rotated about the x-axis.

(6) Find the volume generated when the area bounded by $y = x - x^2$ and $y = 0$ is rotated about the x-axis.

(7) Using integration find the volume generated by rotating the triangle with vertices at $(0, 0)$, $(h, 0)$, and (h, r) about the x-axis.

(8) Using integration find the volume generated by rotating the triangle with vertices at $(0, 0)$, $(h, 0)$, and (h, r) about the y-axis.

(9) A hemispherical bowl of radius a contains water to a depth h.
  (a) Find the volume of water in the bowl.
  (b) Water runs into a hemispherical bowl of radius 5 ft at the rate of 0.2 ft^3/sec. How fast is the water level in the bowl rising when the water is 4 ft deep?

(10) Find the volume generated when the area bounded by $y = -3x - x^2$ and $y = 0$ is rotated about the x-axis.

(11) Find the volume generated when the area bounded by $y = x^2 - 2x$ and $y = 0$ is rotated about the x-axis.

(12) Find the volume generated when the area bounded by $y = x^3$, $x = 2$ and $y = 0$ is rotated about the x-axis.

(13) A football has a volume that is approximately the same as the volume generated by rotating the area inside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (where a and b are constants) about the x-axis. Find the volume generated.

(14) The cross sections of a certain solid by planes perpendicular to the x-axis are circles with diameters extending from the curve $y = x^2$ to the curve $y = 8 - x^2$. The solid lies between the points of intersection of these two curves. Find its volume.

(15) The base of a certain solid is the circle $x^2 + y^2 = a^2$. Each plane section of the solid cut out by a plane perrpendicular to the x-axis is a square with one edge of the square in the base of the solid. Find the volume of the solid.

(16) Find the volume generated when the area bounded by $y = x^4$, $x = 1$ and $y = 0$ is rotated about the x-axis.

(17) Find the volume generated when the area bounded by $y = \sqrt{\cos x}$, $0 \leq x \leq \pi/2$; $x = 0$ and $y = 0$ is rotated about the x-axis.

(18) Find the volume generated when the area bounded by $y = \sqrt{x}$, $y = 2$ and $x = 0$ is rotated about the y-axis.

(19) Two great circles, lying in planes that are perpendicular to each other are marked on a sphere of radius a. A portion of the sphere is shaved off so that any plane section of the remaining solid, perpendicular to the common diameter of the two great circles, is a square with vertices on these circles. Find the volume of the solid that remains.

(20) The base of a solid is the circle $x^2 + y^2 = a^2$. Each plane section of the solid cut out by a plane perpendicular to the y-axis is an isosceles right triangle with one leg in the base of the solid. Find the volume.

(21) The base of a solid is the region between the x-axis and the curve $y = \sin x$ between $x = 0$ and $x = \pi/2$. Each plane section of the solid perpendicular to the x-axis is an equilateral triangle with one side in the base of the solid. Find the volume.

(22) Find the volume generated when the area bounded by $y = \sqrt{x}$, $y = 2$ and $x = 0$ is rotated about the line $y = 2$.

Problem B. Finding volumes by cylindrical shells.

(1) Show that the volume of a right circular cylinder of radius r and height h is $\pi r^2 h$ by using cylindrical shells.

(2) Show that the volume of a right circular cone of radius r and height h is $\frac{1}{3}\pi r^2 h$ by using cylindrical shells.

(3) Show that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ by using cylindrical shells.

(4) A hole of diameter a is bored through the center of a sphere of radius a. Find the remaining volume.

(5) Find the volume of the bagel produced by rotating the circle $x^2 + y^2 = a^2$ about the line $x = b$, $(b \geq a)$.

(6) Find the volume generated by rotating the area bounded by the curves $x + y = 2$, $x = 0$ and $y = 0$ about the x-axis, by using cylindrical shells.

(7) Find the volume generated by rotating the area bounded by the curves $x = 2y - y^2$ and $x = 0$ about the x-axis.

(8) Find the volume generated by rotating the area bounded by the curves $y = 3x - x^2$ and $y = x$ about the x-axis.

(9) Find the volume generated by rotating the area bounded by the curves $y = x$, $y = 1$ and $x = 0$ about the x-axis.

(10) Find the volume generated by rotating the area bounded by the curves $y = x^2$ and $y = 4$ about the x-axis.

(11) Find the volume generated by rotating the area bounded by the curves $y = 3 + x^2$ and $y = 4$ about the x-axis.

(12) Find the volume generated by rotating the area bounded by the curves $y = x^2 + 1$ and $y = x + 3$ about the x-axis.

(13) Find the volume generated by rotating the area bounded by the curves $y = 4 - x^2$ and $y = 2 - x$ about the x-axis.

(14) Find the volume generated by rotating the area bounded by the curves $y = x^4$, $x = 1$ and $y = 0$ about the y-axis.

(15) Find the volume generated by rotating the area bounded by the curves $y = x^3$, $x = 2$ and $y = 0$ about the y-axis.

(16) Find the volume generated by revolving the triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 2)$ about the x-axis.

(17) Find the volume generated by revolving the triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 2)$ about the y-axis.

(18) Find the volume generated by revolving the area bounded by the curves $x = y - y^3$, $x = 0$, $y = 0$ and $y = 1$ about the x-axis.

(19) Find the volume generated by revolving the area bounded by $y = \sqrt{x}$, $x = 0$ and $y = 2$ about the x-axis.

(20) Find the volume generated by revolving the area bounded by $y = \sqrt{x}$, $x = 0$ and $y = 2$ about the line $x = 4$.

Problem C. Practical volumes.

(1) The cross section of a solid in any plane perpendicular to the x-axis is a circle having diameter AB with A on the curve $y^2 = 4x$ and B on the curve $x^2 = 4y$. Find the volume of the solid lying between the points of intersection of the curves.

(2) The base of a solid is the area bounded by $y^2 = 4ax$ and $x = a$. Each cross section perpendicular to the x-axis is an equilateral triangle. Find the volume of the solid.

(3) Find the volume of the slice obtained by cutting a slice off a sphere of radius r, if the slice has thickness h at its thickest point.

(4) Find the volume left after slicing off the top of a right circular cone, if the cone has radius r and height h and, after slicing the top off, what's left has height b.

(5) Find the volume of a tetrahedron, where each side of the tetrahedron is an equilateral triangle with side length a.

(6) The base of a solid is a circle of radius r and the cross sections perpendicular to the base are squares. Find the volume.

(7) The base of a solid is the ellipse $9x^2 + 4y^2 = 36$. Cross sections perpendicular to the x-axis are isosceles right triangles with hypotenuse in the base. Find the volume.

(8) The base of a solid is the parabolic region $\{(x, y) \mid x^2 \leq y \leq 1\}$. Cross sections perpendicular to the y-axis are equilateral triangles. Find the volume.

(9) Find the volume common to two spheres, each with radius r, where the center of each sphere is on the surface of the other.

(10) Find the volume common to two circular cylinders of radius r, such that the axes of the cylinders intersect a right angles.

(11) In 1715 Kepler published a book, *Stereometria doliorum*, which explained how to find the volumes of barrels. A barrel of height h and maximum radius R is constructed by rotating the parabola $y = R - cx^2$, $-h/2 \leq x \leq h/2$, where c is a positive constant.
(a) Show that the radius of each end of the barrel is $r = R - ch^2/4$.
(b) Show that the volume of the barrel is $(1/3)\pi h(2R^2 + r^2 - (1/40)c^2h^4)$.

(12) Suppose that you are given two spherical balls of wood, one of radius r and a second one of radius R. A circular hole is bored through the center of each ball and the resulting napkin rings have the same height h. Which napkin ring contains more wood? How much more?

(13) A right circular cone with height 1 meter and base radius r is to be separated into three pieces of equal volume by cutting twice parallel to the base. At what heights should the cuts be made?

(14) A drinking cup filled with water has the shape of a right circular cone with height h and semivertical angle θ. A ball is placed in the cup displacing some of the water. What is the radius of the ball that causes the greatest volume of water to spill out of the cup?