MATH 221: Calculus and Analytic Geometry
Prof. Ram, Fall 2004

HOMEWORK 12
DUE November 29, 2004

Problem A. Length of a plane curve.

(1) Use integration to show that the circumference of a circle of radius r is $2\pi r$.

(2) Find the length of the curve $y = x^{2/3}$ between $x = -1$ and $x = 8$.

(3) Find the total length of the curve determined by the equations $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

(4) Find the length of the curve $y = (1/3)(x^2+2)^{3/2}$ from $x = 0$ to $x = 3$.

(5) Find the length of the curve $y = x^{3/2}$ from $(0,0)$ to $(4,8)$.

(6) Find the length of the curve $9x^2 = 4y^3$ from $(0,0)$ to $(2\sqrt{3}, 3)$.

(7) Find the length of the curve $y = (1/3)x^3 + 1/4x$ from $x = 1$ to $x = 3$.

(8) Find the length of the curve $x = y^4/4 + 1/8y^2$ from $y = 1$ to $y = 2$.

(9) Find the length of the curve $(y+1)^2 = 4x^3$ from $x = 0$ to $x = 1$.

(10) Find the distance traveled between $t = 0$ and $t = \pi/2$ by a particle $P(x, y)$ whose position at time t is given by $x = a\cos t + at\sin t$, $y = a\sin t - at\cos t$, where a is a positive constant.

(11) Find the length of the curve $x = t - \sin t$, $y = 1 - \cos t$, $0 \le t \le 2\pi$.

(12) Find the distance traveled by the particle $P(x, y)$ between $t = 0$ and $t = 4$ if the position at time t is given by $x = t^2/2$, $y = (1/3)(2t+1)^{3/2}$.

(13) The position of the particle $P(x, y)$ at time t is given by $x = (1/3)(2t+3)^{3/2}$, $y = t^2/2 + t$. Find the distance it travels between $t = 0$ and $t = 3$.

(14) Find the length of the curve $x = (3/5)y^{5/3} - (3/4)y^{1/3}$ from $y = 0$ to $y = 1$.

(15) Find the length of the curve $y = (2/3)x^{3/2} - (1/2)x^{1/2}$ from $x = 0$ to $x = 4$.

(16) Consider the curve $y = f(x)$, $x \geq 0$, such that $f(0) = a$. Let $s(x)$ denote the arc length along the curve from $(0, a)$ to $(x, f(x))$. Find $f(x)$ if $s(x) = Ax$. What are the permissible values of A?

(17) Consider the curve $y = f(x)$, $x \geq 0$, such that $f(0) = a$. Let $s(x)$ denote the arc length along the curve from $(0, a)$ to $(x, f(x))$. Is it possible for $s(x)$ to equal x^n with $n > 1$? Give a reason for your answer.

Problem B. Surface area.

(1) Use integration to show that the surface area of a sphere of radius r is $4\pi r^2$.

(2) Find the surface area of the bagel obtained by rotating the circle $x^2 + y^2 = r^2$ about the line $y = -r$.

(3) Find the surface area of the solid generated by rotating the portion of the curve $y = (1/3)(x^2 + 2)^{3/2}$ between $x = 0$ and $x = 3$ about the x-axis.

(4) Find the area of the surface generated by rotating the arc of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis.

(5) Find the area of the surface generated by rotating the arc of the curve $y = x^2$ between $(0, 0)$ and $(2, 4)$ about the y-axis.

(6) The arc of the curve $y = x^3/3 + 1/4x$ from $x = 1$ to $x = 3$ is rotated about the line $y = -1$. Find the surface area generated.

(7) The arc of the curve $x = y^4/4 + 1/8y^2$ from $y = 1$ to $y = 2$ is rotated about the x-axis. Find the surface area generated.

(8) Find the area of the surface obtained by rotating about the y-axis the curve $y = x^2/2 + 1/2$, $0 \leq x \leq 1$.

(9) Find the area of the surface obtained by rotating the curve determined by $x = a\cos^3\theta$, $y = a\sin^3\theta$ about the x-axis.

(10) The curve described by the particle $P(x, y)$ with position given by $x = t + 1$, $y = t^2/2 + t$, from $t = 0$ to $t = 4$ is rotated about the y-axis. Find the surface area that is generated.

(11) The loop of the curve $9x^2 = y(3 - y)^2$ is rotated about the x-axis. Find the surface area generated.

(12) Find the surface area generated when the curve $y = (2/3)x^{3/2} - (1/2)x^{1/2}$ from $x = 0$ to $x = 4$ is rotated about the y-axis.

(13) Find the surface area generated when the curve $x = (3/5)y^{5/3} - (3/4)y^{1/3}$ from $y = 0$ to $y = 1$ is rotated about the line $y = -1$.

Problem C. Center of mass.

(1) Find the center of mass of a thin homogeneous triangular plate of base b and height h.

(2) A thin homogeneous wire is bent to form a semicircle of radius r. Find its center of mass.

(3) Find the center of mass of a solid hemisphere of radius r if its density at any point P is proportional to the distance between P and the base of the hemisphere.

(4) Find the center of mass of a thin homogeneous plate covering the area in the first quadrant of the circle $x^2 + y^2 = a^2$.

(5) Find the center of mass of a thin homogeneous plate covering the area bounded by the parabola $y = h^2 - x^2$ and the x-axis.

(6) Find the center of mass of a thin homogeneous plate covering the "triangular" area in the first quadrant between the circle $x^2 + y^2 = a^2$ and the lines $x = a$, $y = a$.

(7) Find the center of mass of a thin homogeneous plate covering the area between the x-axis and the curve $y = \sin x$ between $x = 0$ and $x = \pi$.

(8) Find the center of mass of a thin homogeneous plate covering the area between the y-axis and the curve $x = 2y - y^2$.

(9) Find the distance, from the base, of the center of mass of a thin triangular plate of base b and height h if its density varies as the square root of the distance from the base.

(10) Find the distance, from the base, of the center of mass of a thin triangular plate of base b and height h if its density varies as the square of the distance from the base.

(11) Find the center of mass of a homogeneous right circular cone.

(12) Find the center of mass of a solid right circular cone if the density varies as the distance from the base.

(13) A thin homogeneous wire is bent to form a semicircle of radius r. Suppose that the density is $d = k \sin \theta$, where k is a constant. Find the center of mass.

(14) Find the center of gravity of a solid hemisphere of radius r.

(15) Find the center of gravity of a thin hemispherical shell of inner radius r and thickness t.

(16) Find the center of gravity of the area bounded by the x-axis and the curve $y = c^2 - x^2$.

(17) Find the center of gravity of the area bounded by the y-axis and the curve $x = y - y^3$, $0 \leq y \leq 1$.

(18) Find the center of gravity of the area bounded by the curve $y = x^2$ and the line $y = 4$.

(19) Find the center of gravity of the area bounded by the curve $y = x - x^2$ and the line $x + y = 0$.

(20) Find the center of gravity of the area bounded by the curve $x = y^2 - y$ and the line $y = x$.

(21) Find the center of gravity of a solid right circular cone of altitude h and base radius r.

(22) Find the center of gravity of the solid generated by rotating, about the y axis, the area bounded by the curve $y = x^2$ and the line $y = 4$.

(23) The area bounded by the curve $x = y^2 - y$ and the line $y = x$ is rotated about the x axis. Find the center of gravity of the solid thus generated.

(24) Find the center of gravity of a very thin right circular conical shell of base radius r and height h.

(25) Find the center of gravity of the surface area generated by rotating about the line $x = -r$, the arc of the circle $x^2 + y^2 = r^2$ that lies in the first quadrant.

(26) Find the moment, about the x-axis of the arc of the parabola $y = \sqrt{x}$ lying between $(0, 0)$ and $(4, 2)$.

(27) Find the center of gravity of the arc length of one quadrant of a circle.

Problem D. Average value of a function.

(1) Explain how to derive a formula for the average value of a function $f(x)$ as x ranges from a to b.

(2) Compute the average of the numbers $1, 2, 3, \ldots, 100$.

(3) Compute the average of the numbers $9, 10, 11, \ldots, 243$.

(4) Compute the average of the numbers $-9, -6, -3, 0, 3, 6, 9, \ldots, 243$.

(5) Compute the average of the numbers $3^0, 3^1, 3^2, \ldots, 3^{50}$.

(6) Explain why the average of the numbers $1, 1/2, 1/3, \ldots, 1/100$ is more than .04615 but less than .04705.

(7) Explain why the average of the numbers $1, e^{-1}, e^{-2}, \ldots, e^{-50}$ is more than .02 but less than .04.

(8) Show that the average of the numbers $1, e^{-1}, e^{-2}, \ldots, e^{-50}$ is equal to .031639534 (up to 7 decimal places).

(9) Explain why the average of the numbers $1, 1/4, 1/9, 1/16, 1/25, \ldots, 1/10000$ is more than .00333433 but less than .01333333.

(10) Graph $f(x) = \sin x$, $0 \leq x \leq \pi/2$, and find its average value. Indicate the average value on the graph. Draw a rectangle with base $0 \leq x \leq \pi/2$ and with area equal to the area under the graph of $f(x)$.

(11) Graph $f(x) = \sin x$, $0 \leq x \leq 2\pi$, and find its average value. Indicate the average value on the graph. Draw a rectangle with base $0 \leq x \leq 2\pi$ and with area equal to the area under the graph of $f(x)$.

(12) Graph $f(x) = \sin^2 x$, $0 \leq x \leq \pi/2$, and find its average value. Indicate the average value on the graph. Draw a rectangle with base $0 \leq x \leq \pi/2$ and with area equal to the area under the graph of $f(x)$.

(13) Graph $f(x) = \sin^2 x$, $\pi \leq x \leq 2\pi$, and find its average value. Indicate the average value on the graph. Draw a rectangle with base $\pi \leq x \leq 2\pi$ and with area equal to the area under the graph of $f(x)$.

(14) Graph $f(x) = \sqrt{2x+1}$, $4 \leq x \leq 12$, and find its average value. Indicate the average value on the graph. Draw a rectangle with base $4 \leq x \leq 12$ and with area equal to the area under the graph of $f(x)$.

(15) Graph $f(x) = 1/2 + (1/2)\cos 2x$, $0 \leq x \leq \pi$, and find its average value. Indicate the average value on the graph. Draw a rectangle with base $0 \leq x \leq \pi$ and with area equal to the area under the graph of $f(x)$.

(16) Graph $f(x) = \alpha x + \beta$, $a \leq x \leq b$, where α, β, a and b are constants, and find its average value. Draw a rectangle with base $a \leq x \leq b$ and with area equal to the area under the graph of $f(x)$.

(17) A mailorder company receives 600 cases of athletic socks every 60 days. The number of cases on hand t days after the shipment arrives is $I(t) = 600 - 20\sqrt{15t}$. Find the

average daily inventory. If the holding cost for one case is 1/2 cent per day, find the total daily holding cost.

(18) Find the average value of y with respect to x for that part of the curve $y = \sqrt{ax}$ between $x = a$ and $x = 3a$.

(19) Find the average value of y^2 with respect to x for the curve $ay = b\sqrt{a^2 - x^2}$ between $x = 0$ and $x = a$. Also find the average value of y with respect to x^2 for $0 \leq x \leq a$.

(20) A point moves in a straight line during the time from $t = 0$ to $t = 3$ according to the law $s = 120t - 16t^2$.

(a) Find the average value of the velocity, with respect to time, for these three seconds.

(b) Find the average value of the velocity, with respect to the distance s, for these three seconds.

(21) The temperature in a certain city t hours after 9 am was approximated by the function $T(t) = 50 + 14\sin(\pi t/12)$. Find the average temperature during the period from 9 am to 9 pm.