MATH 221: Calculus and Analytic Geometry
Prof. Ram, Fall 2004

HOMEWORK 7
DUE October 25, 2004

For each of the following graphing problems also determine

(a) where $f(x)$ is defined,
(b) where $f(x)$ is continuous,
(c) where $f(x)$ is differentiable,
(d) where $f(x)$ is increasing and where it is decreasing,
(e) where $f(x)$ is concave up and where it is concave down,
(f) what the critical points of $f(x)$ are,
(g) where the points of inflection are, and
(h) what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

Problem A. Graphing rational functions.

(1) Graph $f(x) = 1/x$.

(2) Graph the function $f(x)$ such that $\frac{df}{dx} = 1/x$ and $f(-1) = 2$ and $f(1) = 1$.

(3) Graph $f(x) = x + 1/x$.

(4) Graph $f(x) = \frac{x^2 + 2x - 20}{x - 4}$.

(5) Graph $f(x) = \frac{1}{x^2 + 1}$.

(6) Graph $f(x) = \frac{1}{x^2 + 2x + c}$, where c is a constant.

(7) Graph $f(x) = \frac{x^3}{x^2 + 1}$.

(8) Graph $f(x) = \frac{x^2 - 1}{x^2 + 1}$.

(9) Graph $f(x) = \frac{2x^2}{x^2 - 1}$.

(10) Graph $f(x) = \dfrac{x^2 + 7x + 3}{x^2}$.

(11) Graph $f(x) = \dfrac{x^2(x+1)^3}{(x-2)^2(x-4)^4}$.

(12) Graph $f(x) = \dfrac{x^2 - 1}{x^3 - 4x}$.

Problem B. Graphing functions with square roots.

(1) Graph $y = f(x)$ where $x^2 + y^2 = 1$.

(2) Graph $f(x) = \sqrt{1 - x^2}$.

(3) Graph $f(x) = \sqrt{a^2 - x^2}$, where a is a constant.

(4) Graph $y = f(x)$ when $(x-h)^2 + (y-k)^2 = r^2$, where h, k, and r are constants.

(5) Graph $y = f(x)$ when $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 = r^2$, where h, k, and r are constants.

(6) Graph $y = f(x)$ when $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where a and b are constants.

(7) Graph $y = f(x)$ when $x = a\cos\theta$ and $y = b\sin\theta$, where a and b are constants.

(8) Graph $f(x) = (b/a)\sqrt{a^2 - x^2}$, where a and b are constants.

(9) Graph $y = f(x)$ when $x^2 - y^2 = 1$.

(10) Graph $y = f(x)$ when $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where a and b are constants.

(11) Graph $y = f(x)$ when $y = ax^2 - b$, where a and b are constants.

(12) Graph $y = f(x)$ when $x = 2y^2 - 1$.

(13) Graph $y = f(x)$ when $x = \cos 2\theta$ and $y = \cos\theta$.

(14) Graph $f(x) = b\sqrt{x - a}$, where a and b are constants.

(15) Graph $f(x) = \sqrt{x + 2}$.

(16) Graph $f(x) = -\sqrt{x + 2}$.

(17) Graph $y = f(x)$ when $y^2(x^2 - x) = x^2 - 1$.

(18) Graph $y = f(x)$ when $x = \dfrac{y^2 - 1}{y^2 + 1}$.

(19) Graph $y = f(x)$ when $y = \dfrac{\sqrt{1 + x}}{\sqrt{1 - x}}$.

(20) Graph $f(x) = \dfrac{x^2}{\sqrt{x + 1}}$.

(21) Graph $f(x) = x\sqrt{32 - x^2}$.

(22) Graph $f(x) = x\sqrt{1 - x^2}$.

Problem C. Graphing other functions.

(1) Graph $f(x) = \lfloor x \rfloor$.

(2) Graph $f(x) = |x|$.

(3) Graph $f(x) = |x - 5|$.

(4) Graph $f(x) = |x^2 - 1|$.

(5) Graph $f(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$

(6) Graph $f(x) = (x - 1)^{1/3}$.

(7) Graph $f(x) = x^{2/3}$.

(8) Graph $f(x) = \dfrac{1}{(x - 1)^{2/3}}$.

(9) Graph $f(x) = x(1 - x)^{2/5}$.

(10) Graph $f(x) = x^{2/3}(6 - x)^{1/3}$.

(11) Graph $y = f(x)$ when $\sqrt{x} + \sqrt{y} = 1$.

(12) Graph $y = f(x)$ when $x^{2/3} + y^{2/3} = a^{2/3}$.

(13) Graph $y = f(x)$ when $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

(14) Graph $f(x) = \sin x$.

(15) Graph $f(x) = \sin 2x - x$.

(16) Graph $f(x) = \sin x - \cos x$ for $-\pi/3 < x < 0$.

(17) Graph $f(x) = 2\cos x + \sin 2x$.

(18) Graph $f(x) = \dfrac{\sin x}{x}$.

(19) Graph $f(x) = \sin(1/x)$.

(20) Graph $f(x) = \sin(x + \sin 2x)$.

(21) Graph $f(x) = e^{-x}$.

(22) Graph $f(x) = e^{1/x}$.

(23) Graph $f(x) = e^{-x^2}$.

(24) Graph $f(x) = \ln(4 - x^2)$.

Problem D. Rolle's theorem and the mean value theorem.

(1) State Rolle's theorem and draw a picture which illustrates the statement of the theorem.

(2) State the mean value theorem and draw a picture which illustrates the statement of the theorem.

(3) Explain why Rolle's theorem is a *special case* of the mean value theorem.

(4) Verify Rolle's theorem for the function $f(x) = (x-1)(x-2)(x-3)$ on the interval $[1, 3]$.

(5) Verify Rolle's theorem for the function $f(x) = (x-2)^2(x-3)^6$ on the interval $[2, 3]$.

(6) Verify Rolle's theorem for the function $f(x) = \sin x - 1$ on the interval $[\pi/2, 5\pi/2]$.

(7) Verify Rolle's theorem for the function $f(x) = e^{-x}\sin x$ on the interval $[0, \pi]$.

(8) Verify Rolle's theorem for the function $f(x) = x^3 - 6x^2 + 11x - 6$.

(9) Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but that there is no number c in the interval $(-1, 1)$ such that $\left.\dfrac{df}{dx}\right|_{x=c} = 0$. Why does this not contradict Rolle's theorem?

(10) Let $f(x) = (x - 1)^{-2}$. Show that $f(0) = f(2)$ but that there is no number c in the interval $(0, 2)$ such that $\left.\dfrac{df}{dx}\right|_{x=c} = 0$. Why does this not contradict Rolle's theorem?

(11) Discuss the applicability of Rolle's theorem when $f(x) = (x-1)(2x-3)$ on the interval $1 \leq x \leq 3$.

(12) Discuss the applicability of Rolle's theorem when $f(x) = 2+(x-1)^{2/3}$ on the interval $0 \leq x \leq 2$.

(13) Discuss the applicability of Rolle's theorem when $f(x) = \lfloor x \rfloor$ on the interval $-1 \leq x \leq 1$.

(14) At what point on the curve $y = 6 - (x - 3)^2$ on the interval $[0, 6]$ is the tangent to the curve parallel to the x-axis?

(15) Show that the equation $x^5 + 10x + 3 = 0$ has exactly one real root.

(16) Show that a polynomial of degree three has at most three real roots.

(17) Verify the mean value theorem for the function $f(x) = x^{2/3}$ in the interval $[0, 1]$.

(18) Verify the mean value theorem for the function $f(x) = \ln x$ in the interval $[1, e]$.

(19) Verify the mean value theorem for the function $f(x) = x$ in the interval $[a, b]$.

(20) Verify the mean value theorem for the function $f(x) = \ell x^2 + mx + n$ in the interval $[a, b]$, where ℓ, m and n are constants.

(21) Show that the mean value theorem is not applicable to the function $f(x) = |x|$ in the interval $[-1, 1]$.

(22) Show that the mean value theorem is not applicable to the function $f(x) = 1/x$ in the interval $[-1, 1]$.

(23) Find the points on the curve $y = x^3 - 3x$ where the tangent is parallel to the chord joining $(1, -2)$ and $(2, 2)$.

(24) If $f(x) = x(1 - \ln x)$, $x > 0$, show that $(a - b) \ln c = b(1 - \ln b) - a(1 - \ln a)$, where $0 < a < b$.

Problem E. Tangents and normals.

(1) Find the slope of the tangent to the curve $y = x^3 - x$ at $x = 2$.

(2) Find the slope of the tangent to the curve $y = (\sin 2x + \cot x + 2)^2$ at $x = \pi/2$.

(3) Find the equations of the tangent and normal to the curve $y = x^3 - 2x + 7$ at the point $(1, 6)$.

(4) Find the equations of the tangent and normal to the curve $3xy^2 - 2x^2y = 1$ at the point $(1, 1)$.

(5) Find the equations of the tangent and normal to the curve $y = x^3 + 2x + 6$ at the point $(2, 18)$.

(6) Find the equations of the tangent and normal to the curve $y^2 = 4ax$ at the point $(a/m^2, 2a/m)$.

(7) Find the equations of the tangent and normal to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a \cos \theta, b \sin \theta)$.

(8) Find the equations of the tangent and normal to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a \sec \theta, b \tan \theta)$.

(9) Find the equations of the tangent and normal to the curve $c^2(x^2 + y^2) = x^2y^2$ at the point $(c/\cos \theta, c/\sin \theta)$.

(10) Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

(11) Find the equation of the normal to the curve $ay^2 = x^3$ at the point $(am^2, 2m^3)$.

(12) Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (p, q) is $\frac{xp}{a^2} - \frac{yq}{b^2} = 1$