Math 521 Lecture 3: Homework 3
Fall 2004, Professor Ram
Due September 27, 2004

1 Favourites

An arsenal of examples in your head is crucial to processing mathematical concepts. For each of the following, list your favourite examples. Make sure your list includes enough examples to develop an understanding of the concept. If it is not clear that your example is an example then prove that it is.

1. sets
2. functions
3. non-functions
4. relations
5. operations
6. monoids
7. groups
8. rings
9. fields
10. division rings
11. integral domains
12. fields of fractions
13. polynomials
14. formal power series
15. equivalence relations
16. rational numbers
17. irrational numbers
18. algebraic numbers

19. transcendental numbers
20. vector spaces
21. algebras
22. derivations

2 Exercises

1. Let A be a ring and let $a \in A$. Define 0, $-a$ and a^{-1} and show that $0 \cdot a = 0$, $0 + 0 = 0$, $-(-a) = a$, $(-1)a = -a$, and $(a^{-1})^{-1} = a$.
2. Show that a finite integral domain is a field.
3. Explain how long division works for polynomials and give some examples.
4. Explain why it is necessary to assume that A is an integral domain when constructing the field of fractions of A.
5. Show that the addition operation in the field of fractions is well defined.
6. Show that the multiplication operation in the field of fractions is well defined.
7. Show that the field of fractions is a field.
8. Let A be an integral domain and let $\mathbb{F}$ be the field of fractions of $\mathbb{F}$. Show that the map
$$\begin{array}{rccc} \iota\colon & A & \longrightarrow & \mathbb{F} \\ & a & \longmapsto & \frac{a}{1} \end{array}$$
is an injective ring homomorphism.
9. Let A be an integral domain and let $\mathbb{F}$ be the field of fractions of $\mathbb{F}$. Show that if $\mathbb{K}$ is a field with an injective ring homomorphism $\zeta\colon A \to \mathbb{K}$ then there is a unique ring homomorphism $\varphi\colon \mathbb{F} \to \mathbb{K}$ such that $\zeta = \varphi \circ \iota$.
10. Let $\mathbb{F}$ be a field and let $a \in \mathbb{F}$. The evaluation homomorphism $\mathrm{ev}_a\colon \mathbb{F}[x] \to \mathbb{F}$ is defined from $\mathbb{F}[x]$ to $\mathbb{F}$. Discuss thoroughly the issue of extending the evaluation homomorphism ev_a to $\mathbb{F}[[x]]$, $\mathbb{F}(x)$, and $\mathbb{F}((x))$.
11. Let S be a set of cardinality n. Show that $\binom{n}{k}$ is the number of subsets of S of cardinality k.
12. Show that $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in $(x+y)^n$.
13. Show that $\binom{n}{n} = 1$, $\binom{n}{0} = 1$ and, if $1 \le k \le n-1$, then
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

14. Define e^x in 6 different ways and prove that all 6 definitions are equivalent.

15. Define $\ln x$ in at least 3 different ways and prove that your definitions are equivalent.

16. Define $\sin x$ in at least 3 ways and prove that your definitions match each other.

17. Explain why $\frac{1}{1-x} = 1+x+x^2+x^3+\cdots$. What does $\frac{1}{1-x}$ mean? Where do these expressions live?

18. Let $n \in \mathbb{Z}_{>0}$. Explain why $\dfrac{x^n-1}{x-1} = 1+x+x^2+x^3+\cdots+x^{n-1}$. What does $\dfrac{x^n-1}{x-1}$ mean? Where do these expressions live.

19. Let $n \in \mathbb{Z}_{>0}$. Define $\dbinom{n}{k}$. Prove that $(x+y)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

20. Define e^x and prove that $e^x e^y = e^{(x+y)}$. Where do these expressions live?

21. Let $G = \{p(x) \in \mathbb{F}[[x]] \mid p(0) = 1\}$. Show that G is an abelian group under multiplication.

22. Let $\mathfrak{g} = \{p(x) \in \mathbb{F}[[x]] \mid p(0) = 0\}$. Show that $\mathfrak{g}$ is an abelian group under addition.

23. Let $G = \{p(x) \in \mathbb{F}[[x]] \mid p(0) = 1\}$ and $\mathfrak{g} = \{p(x) \in \mathbb{F}[[x]] \mid p(0) = 0\}$. Show that

$$\begin{array}{ccc} G & \longrightarrow & \mathfrak{g} \\ p & \longmapsto & e^p - 1 \end{array}$$

is an isomorphism of groups.

24. Show that $\ln(1+(e^x-1)) = e^{\ln(1+x)} - 1 = x$.

25. Show that there is a unique derivation $\dfrac{d}{dx}$ of $\mathbb{F}[x]$ such that $\dfrac{dx}{dx} = 1$.

26. Show that if $p \in \mathbb{F}[x]$ then

$$\frac{dp}{dx} = (\text{coefficient of } y \text{ in } p(x+y)).$$

27. Show that if $p \in \mathbb{F}[x]$ then

$$p = \sum_{k \in \mathbb{Z}_{\geq 0}} \left(\left(\frac{d}{dx}\right)^k p \right)(0)\; x^k.$$

28. Show that there is a unique extension of $\dfrac{d}{dx}$ to a derivation of $\mathbb{F}(x)$.

29. Show that there is a unique extension of $\dfrac{d}{dx}$ to a derivation of $\mathbb{F}[[x]]$.

30. Show that there is a unique extension of $\dfrac{d}{dx}$ to a derivation of $\mathbb{F}((x))$.

31. Show that if $p \in \mathbb{F}[[x]]$ then

$$\frac{dp}{dx} = (\text{coefficient of } y \text{ in } p(x+y)).$$

32. Show that if $p \in \mathbb{F}[[x]]$ then

$$p = \sum_{k \in \mathbb{Z}_{\geq 0}} \left(\left(\frac{d}{dx} \right)^k p \right) (0)\ x^k.$$

33. Show that if $p \in \mathbb{F}[[x]]$ then

$$\frac{dp}{dx} = \mathrm{ev}_{h=0} \left(\frac{p(x+h) - p(x)}{h} \right)$$

3 Vocabulary

Define the following terms.

1. commutative ring
2. integral domain
3. field of fractions
4. e^x
5. $\ln x$
6. $\sin x$
7. $\cos x$
8. derivation