

Math 521 Lecture 3: Homework 6
Fall 2004, Professor Ram
Due October 25, 2004

1 Favourites

An arsenal of examples in your head is crucial to processing mathematical concepts. For each of the following, list your favourite examples. Make sure your list includes enough examples to develop an understanding of the concept. If it is not clear that your example is an example then prove that it is.

1. filters
2. Hausdorff spaces
3. non Hausdorff spaces
4. compact sets
5. non compact set
6. interior
7. closure
8. limit points
9. sequences
10. series

2 Exercises

1. Let X be a topological space and let $E \subseteq X$. Show that the interior E° of E is the set of interior points of E .
2. Let X be a topological space and let $E \subseteq X$. Show that the closure \overline{E} of E is $E \cup E'$ where E' is the set of limit points of E .
3. Show that every metric space is Hausdorff. Why do we care about this?
4. Let X be a set and let \mathcal{B} be a collection of subsets of X . Show that the collection

$$\mathcal{F} = \{\text{subsets of } X \text{ that contain a set of } \mathcal{B}\},$$

is a filter if and only if

- (a) The intersection of two sets of \mathcal{B} contains a set of \mathcal{B} , and
 (b) \mathcal{B} is not empty, and the empty set is not in \mathcal{B} .
5. Give an example that illustrates the difference between limit points and cluster points.
6. Give an example of a filter that has more than one limit point.
7. Let X be a topological space. Show that the following are equivalent:
- (a) Any two distinct points of X have disjoint neighborhoods.
 (b) The intersection of the closed neighborhoods of any point of X consist of that point alone.
 (c) The diagonal of the product space $X \times X$ is a closed set.
 (d) For every set I , the diagonal of the product space $Y = X^I$ is closed in Y .
 (e) No filter on X has more than one limit point.
 (f) If a filter \mathcal{F} on X converges to x then x is the only cluster point of x .
8. Let X be a topological space. Show that the following are equivalent:
- (a) (a) Every filter on X has at least one cluster point.
 (b) (b) Every ultrafilter is convergent
 (c) (c) Every family of closed subsets of X whose intersection is empty contains a finite subfamily whose intersection is empty.
 (d) (d) Every open cover of X contains a finite subcover.
9. Define
- $$\lim_{x \rightarrow a} f(x) \quad \text{and} \quad \lim_{n \rightarrow \infty} x_n.$$
10. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be continuous functions. Show that $g \circ f$ is a continuous function.
11. Let X and Y be topological spaces and show that a function $f: X \rightarrow Y$ is continuous at a if and only if $\lim_{x \rightarrow a} f(x) = f(a)$.
12. Let X and Y be metric spaces. Show that a function $f: X \rightarrow Y$ is continuous at a if and only if given $\delta \in \mathbb{R}_{>0}$ there exists a $\varepsilon \in \mathbb{R}_{>0}$ such that if $d(x, a) < \delta$ then $d(f(x), f(a)) < \varepsilon$.
13. What is Baby Rudin's definition of a sequence? What is Baby Rudin's definition of a series?
14. What is Baby Rudin's definition for a sequence (x_1, x_2, \dots) to converge to a . How does this compare to definitions of limits by filters?
15. What is Baby Rudin's definition of the lim sup and the lim inf of a sequence (x_1, x_2, \dots) ?
16. What is Baby Rudin's definition for a series $\sum_{n \in \mathbb{Z}_{>0}} x_n$ to converge to a ? What is Baby Rudin's definition for a series $\sum_{n \in \mathbb{Z}_{>0}} x_n$ to converge absolutely to a ?
17. Show that e is irrational.

18. Why do we care about absolute convergence?

Let $\alpha, \beta \in \mathbb{C}$. Determine whether or not the following converge and find the limit.

19. $\lim_{n \rightarrow \infty} \frac{1}{n}$,

20. $\lim_{n \rightarrow \infty} n^2$,

21. $\lim_{n \rightarrow \infty} 1 + \frac{(-1)^n}{n}$,

22. $\lim_{n \rightarrow \infty} 1$,

23. $\lim_{n \rightarrow \infty} \left(\frac{1}{n}\right)^\alpha$,

24. $\lim_{n \rightarrow \infty} \alpha^{1/n}$,

25. $\lim_{n \rightarrow \infty} n^{1/n}$,

26. $\lim_{n \rightarrow \infty} \frac{n^\alpha}{(1+\beta)^n}$,

27. $\lim_{n \rightarrow \infty} \alpha^n$,

28. $\lim_{n \rightarrow \infty} n^{1/n}$,

29. $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{1/n}$,

30. $\sum_{n \in \mathbb{Z}_{>0}} \frac{1}{n}$,

31. $\sum_{n \in \mathbb{Z}_{\geq 0}} \alpha^n$,

32. $\sum_{n \in \mathbb{Z}_{>0}} n^\alpha$,

33. $\sum_{n \in \mathbb{Z}_{>0}} \left(\frac{1}{n}\right)^\alpha$,

34. $\sum_{n \in \mathbb{Z}_{>0}} \left(\frac{1}{n \ln n}\right)^\alpha$,

35. $\sum_{n \in \mathbb{Z}_{>0}} \frac{1}{n!}$,

36. Do Chapter 3, Problem 1 in baby Rudin.

37. Do Chapter 3, Problem 2 in baby Rudin.

38. Do Chapter 3, Problem 3 in baby Rudin.

3 Vocabulary

Define the following terms.

1. $\lim_{x \rightarrow a} f(x)$
2. $\lim_{n \rightarrow \infty} x_n$
3. \limsup
4. \liminf
5. sequence
6. series
7. converge
8. converge absolutely
9. continuous
10. continuous at a
11. Hausdorff
12. filter
13. ultrafilter
14. limit point
15. cluster point
16. neighborhood
17. neighborhood filter
18. Fréchet filter
19. base