Math 521 Lecture 3: Homework 7
Fall 2004, Professor Ram
Due November 1, 2004

1 Exercises

1. Define the terms in, and prove the following theorem:

   Theorem 1.1. *Let X be a topological space and let $E \subseteq X$.*

   (a) The interior E° of E is the set of interior points of E.
   (b) The closure $\overline{E}$ of E is the set of close points of E.

2. Define the terms in, and prove, the following theorem:

   Theorem 1.2. *Let X and Y be topological spaces. A function $f\colon X \to Y$ is continuous at a if and only if $\lim_{x\to a} f(x) = f(a)$.*

3. Define the terms in, and prove, the following theorem:

   Theorem 1.3. *Let X be a topological space and let $(x_1, x_2, \ldots)$ be a sequence in X. Then*

   (a) y is a limit point of $(x_1, x_2, \ldots)$ if and only if, if N_y is a neighborhood of y then there exists $n_0 \in \mathbb{Z}_{>0}$ such that $x_n \in N_x$ for all $n \in \mathbb{Z}_{\geq 0}$, $n \geq n_0$.
   (b) y is a cluster point of $(x_1, x_2, \ldots)$ if and only if, if N_y is a neighborhood of y and $n_0 \in \mathbb{Z}_{>0}$ then there exists $n \in \mathbb{Z}_{>0}$ with $n \geq n_0$ such that $x_n \in N_y$.

4. Define the terms in, and prove, the following theorem:

   Theorem 1.4. *Let X be a topological space. The following are equivalent:*

   (a) Any two distinct points of X have disjoint neighborhoods.
   (b) The intersection of the closed neighborhoods of any point of X consist of that point alone.
   (c) The diagonal of the product space $X \times X$ is a closed set.
   (d) For every set I, the diagonal of the product space $Y = X^I$ is closed in Y.
   (e) No filter on X has more than one limit point.
   (f) If a filter $\mathcal{F}$ on X converges to x then x is the only cluster point of x.

5. Do Chapter 2 problem 22 in baby Rudin.

6. Do Chapter 2 problem 23 in baby Rudin.
7. Do Chapter 2 problem 24 in baby Rudin.
8. Do Chapter 2 problem 25 in baby Rudin.
9. Do Chapter 2 problem 26 in baby Rudin.
10. Do Chapter 4 problem 1 in baby Rudin.
11. Do Chapter 4 problem 2 in baby Rudin.
12. Do Chapter 4 problem 3 in baby Rudin.
13. Do Chapter 4 problem 4 in baby Rudin.
14. Do Chapter 4 problem 5 in baby Rudin.
15. Do Chapter 4 problem 6 in baby Rudin.
16. Define the terms in, and prove, the following theorem:

    Theorem 1.5. *Let X be a topological space. The following are equivalent.*

    (a) Every filter on X has at least one cluster point.
    (b) Every ultrafilter on X is convergent.
    (c) Every family of closed subsets of X whose intersection is empty contains a finite subfamily whose intersection is empty.
    (d) Every open cover of X contains a finite subcover.

17. Define the terms in, and prove, the following theorem:

    Theorem 1.6. *Let X be a metric space and let E be a subset of X. The set E is compact if and only if every infinite subset of E has a limit point in E.*

18. Define the terms in, and prove, the following theorem:

    Theorem 1.7. *Let X be a Hausdorff topological space and let K be a compact subset of X. Then K is closed.*

19. Define the terms in, and prove, the following theorem:

    Theorem 1.8. *(a) A k-cell is compact.*

    (b) Let E be a subset of $\mathbb{R}^k$. If E is closed and bounded then E is compact.