Math 521 Lecture 3: Homework 8
Fall 2004, Professor Ram
Due December 1, 2004

1 Exercises

1. Let X be a metric space and let $E \subseteq X$. Define the *diameter* of E and show that $\mathrm{diam}(\overline{E}) = \mathrm{diam}(E)$.

2. Let X be a metric space. Let K_n be a sequence of compact sets in X such that $K \supseteq K_{n+1}$. Show that
$$\text{if} \quad \lim_{n\to\infty} \mathrm{diam}(K_n) = 0 \quad \text{then} \quad \bigcap_{n=1}^{\infty} K_n$$
consists of exactly one point.

3. Let X be a compact metric space. Show that X is complete.

4. Show that $\mathbb{R}^k$ is complete.

5. Give an example of a metric space that is not complete.

6. Give an example of
   (a) a Cauchy sequence
   (b) A non Cauchy sequence
   (c) A convergent sequence
   (d) a nonconvergent sequence
   (e) a convergent sequence that is Cauchy
   (f) a convergent sequence that is not Cauchy
   (g) a Cauchy sequence that is convergent
   (h) a Cauchy sequence that is not convergent
   (i) a sequence that is non convergent and non Cauchy.

7. Let Xbe a complete metric space. Let E be a closed subset of X. Show that E is complete.

8. Do Chapter 3, problem 20 of Baby Rudin.

9. Do Chapter 3, problem 21 of Baby Rudin.

10. Do Chapter 3, problem 22 of Baby Rudin.

11. Do Chapter 3, problem 23 of Baby Rudin.
12. Do Chapter 3, problem 24 of Baby Rudin.
13. Do Chapter 3, problem 25 of Baby Rudin.
14. Define uniformity, uniform space and entourage.
15. Define the topology on a uniform space. Prove that it exists and is unique.
16. Define Cauchy filter and convergent filter and complete space.
17. Prove that every convergent filter is Cauchy.
18. Give an example of a Cauchy filter that is not convergent.
19. Define uniformly continuous function.
20. Carefully state and prove that every uniformly continuous function is continuous.
21. Give an example which shows that two different uniformities on a set X can give rise to the same topology on X.
22. Carefully state and prove that that if $f\colon X \to Y$ is a uniformly continuous function then f sends Cauchy sequences to Cauchy sequences.
23. If $f\colon X \to Y$ sends Cauchy sequences to Cauchy sequences then is f uniformly continuous?
24. Let X be a metric space. Carefully define the metric space uniformity on X and explain how a Cauchy sequence corresponds to a Cauchy filter.
25. Let X and Y be metric spaces. Explain how the definition of a uniformly continuous function $f\colon X \to Y$ and the definition of a uniformly continuous function between uniform spaces match up.
26. Carefully define completion.
27. Let X be a metric space. Show that the completion of X exists.
28. Let X be a metric space. Show that the completion of X is unique.
29. Let X be a uniform space. Show that the completion of X exists.
30. Let X be a uniform space. Show that the completion of X is unique.
31. Let X be a metric space and let $\hat{X}$ be the completion of X. Show that X is a dense subset of $\hat{X}$.
32. Let X be a uniform space and let $\hat{X}$ be the completion of X. Show that X is a dense subset of $\hat{X}$.
33. Let R be an abelian topological group. Explain how R is a uniform space.
34. Let R be an abelian topological group. Explain how the completion of R can be made into a topological group.
35. Explain how $\mathbb{R}$ is a completion of $\mathbb{Q}$.

36. Explain what the p-adic numbers are.

37. Explain how $\mathbb{F}[[x]]$ is a completion of $\mathbb{F}[x]$.

38. Define carefully $\lim_{x \to p} f(x)$ following Baby Rudin. Pay special attention to the domain and range of f and the location of the point p.

39. In the definition of the derivative of f at a, where is a located? Define the derivative carefully.

40. Define the functions $f(x) = c$, $f(x) = x$, $f(x) = x^2$, $f(x) = x^n$, $f(x) = e^x$ and $f(x) = \sin x$ carefully, in proof machine language.

41. State and prove the sum rule for derivatives.

42. State and prove the product rule for derivatives.

43. State and prove the quotient rule.

44. State and prove the chain rule.

45. State and prove a precise theorem to the effect that if a function is differentiable at a then it is continuous at a.

46. Give two examples of functions $f : \mathbb{R} \to \mathbb{R}$ which are continuous at 0 but not differentiable at 0.

47. Graph carefully the functions $f(x) = \sin x$, $f(x) = \sin^2 x$, $f(x) = \sin(1/x)$, $f(x) = (1/x)\sin x$, $f(x) = x \sin x$, $f(x) = x^2 \sin x$, $x^2 \sin 1/x$, and $f(x) = (1/x)\sin(1/x)$ and explain how you get these graphs.

48. Carefully state and prove Rolle's theorem.

49. Carefully state and prove the mean value theorem.

50. Carefully state and prove the generalized mean value theorem.

51. Give a counterexample to the mean value theorem.

52. Carefully state and prove Taylor's theorem.

53. Carefully state and prove three different theorems that could be called L'Hopital's rule.

54. Give a counter example to L'Hopital's rule.

55. Carefully graph the function $f(x) = e^{ix}$ and explain how you get this graph.

56. Let X be a metric space. Carefully define $\lim_{t \to a} f(t)$.

57. Let X be a metric space. Show that if $\lim_{t \to a} f(t)$ exists then it is unique.

58. Let X be a topological space. Carefully define $\lim_{t \to a} f(t)$.

59. Let X be a topological space. Show that if f takes values in a Hausdorff space and $\lim_{t \to a} f(t)$ exists then it is unique.

60. Let X be a set. Carefully define $\lim_{\mathcal{F}} f$.

61. Let X be a set. Show that if f takes values in a Hausdorff topological space and $\lim_{\mathcal{F}} f$ exists then it is unique.

62. Define topological group, topological ring, topological division ring, and topological field.

63. Carefully define $f+g$, fg, $-f$, $(1/f)$ and f/g and $f \circ g$.

64. Let X be a metric space. Carefully state and prove that

$$\lim_{t\to a}(f+g)(t) = \lim_{t\to a} f(t) + \lim_{t\to a} g(t),$$

for real valued functions f and g.

65. Let X be a metric space. Carefully state and prove that

$$\lim_{t\to a}(fg)(t) = \big(\lim_{t\to a} f(t)\big)\big(\lim_{t\to a} g(t)\big),$$

for real valued functions f and g.

66. Let X be a metric space. Carefully state and prove that

$$\lim_{t\to a}(-f)(t) = -\lim_{t\to a} f(t).$$

for real valued functions f.

67. Let X be a metric space. Carefully state and prove that

$$\lim_{t\to a}(1/f)(t) = \frac{1}{\lim_{t\to a} f(t)}.$$

for real valued functions f.

68. Let X be a topological space. Carefully state and prove that

$$\lim_{t\to a}(fg)(t) = \big(\lim_{t\to a} f(t)\big)\big(\lim_{t\to a} g(t)\big),$$

for functions f and g with values in a Hausdorff topological group G.

69. Let X be a topological space. Carefully state and prove that

$$\lim_{t\to a}(1/f)(t) = \frac{1}{\lim_{t\to a} f(t)}$$

for functions f and g with values in a Hausdorff topological group G.

70. Let X be a topological space. Carefully state and prove that

$$\lim_{t\to a}(f+g)(t) = \lim_{t\to a} f(t) + \lim_{t\to a} g(t),$$

for functions f and g with values in a Hausdorff topological ring R.

71. Let X be a set. Carefully state and prove that

$$\lim_{\mathcal{F}} fg = \big(\lim_{\mathcal{F}} f\big)\big(\lim_{\mathcal{F}} g\big),$$

for functions f and g with values in a Hausdorff topological group G.

72. Let X be a set. Carefully state and prove that

$$\lim_{\mathcal{F}}(1/f) = \frac{1}{\lim_{\mathcal{F}} f}$$

for a function f with values in a Hausdorff topological group G.

73. Let X be a set. Carefully state and prove that

$$\lim_{\mathcal{F}}(f+g) = \lim_{\mathcal{F}} f + \lim_{\mathcal{F}} g,$$

for functions f and g with values in a Hausdorff topological ring R.