Math 521: Lecture Notes, Fall 2004

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1 Numbers

The **positive integers** is the set

$$\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\} \qquad \text{with the operation} \qquad \begin{array}{rcl} \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} & \to & \mathbb{Z}_{>0} \\ (i,j) & \mapsto & i+j \end{array}$$

The **nonnegative integers** is the set

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\} \qquad \text{with the operation} \qquad \begin{array}{rcl} \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} & \to & \mathbb{Z}_{\geq 0} \\ (i,j) & \mapsto & i+j \end{array}$$

The advantage of the nonnegative integers $\mathbb{Z}_{\geq 0}$ over the positive integers $\mathbb{Z}_{>0}$ is that $\mathbb{Z}_{\geq 0}$ contains an identity element for the operation and $\mathbb{Z}_{>0}$ does not.

The **integers** is the set

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

with the operations

$$\begin{array}{rcl} \mathbb{Z} \times \mathbb{Z} & \to & \mathbb{Z} \\ (i,j) & \mapsto & i+j \end{array}$$

The advantage of the integers $\mathbb{Z}$ over the nonnegative integers $\mathbb{Z}_{\geq 0}$ is that every element of $\mathbb{Z}$ has an inverse; this is not true in $\mathbb{Z}_{\geq 0}$. There is another operation on $\mathbb{Z}$,

$$\begin{array}{rcl} \mathbb{Z} \times \mathbb{Z} & \to & \mathbb{Z} \\ (i,j) & \mapsto & i+(-j) \end{array}$$

but this operation is not very well behaved: it is not associative and not commutative (though it does have an identity). There is another operation on $\mathbb{Z}$

$$\begin{array}{rcl} \mathbb{Z} \times \mathbb{Z} & \to & \mathbb{Z} \\ (i,j) & \mapsto & ij \end{array}$$

and this operation is associative, commutative and has an identity but does not have inverses.

The **rationals** is the set

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\} \qquad \text{where} \qquad \frac{a}{b} = \frac{c}{d} \qquad \text{if } ad = bc,$$

with operations defined by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \qquad \text{and} \qquad \frac{a}{b}\,\frac{c}{d} = \frac{ac}{bd}.$$

The advantage of the rationals $\mathbb{Q}$ over the integers $\mathbb{Z}$ is that the multiplication has inverses; well, ... almost has inverses–the element 0 does not have an inverse.

By long division, every rational number $\frac{a}{b}$ can be represented as a decimal expansion

$$d_r d_{r-1} \cdots d_1 d_0 . d_{-1} d_{-2} d_{-3} \cdots$$

where the idea is that

$$d_r d_{r-1} \cdots d_1 d_0 . d_{-1} d_{-2} d_{-3} \cdots = \sum_{\ell \in \mathbb{Z}, \ell \le r} d_\ell 10^\ell .$$

If $a = a_r \ldots a_1 a_0 . a_{-1} a_{-2} \ldots$ is a decimal expansion let $a_{\le n}$ be the element of $\mathbb{Q}$ given by

$$a_{\le n} = a_r \ldots a_1 a_0 . a_{-1} a_{-2} \ldots a_{-(n-1)} a_{-n}.$$

The **real numbers** is the set $\mathbb{R}$ of decimal expansions

$$\mathbb{R} = \{ \ d_r \cdots d_1 d_0 . d_{-1} d_{-2} \cdots \ \mid d_i \in \{0,1,2,\ldots,9\} \ \}$$

with

$$a = b \quad \text{for all } n \in \mathbb{Z}_{>0} \ (a_{\le n} - b_{\le n})_{\le n-1} = 0 \text{ in } \mathbb{Q},$$

and operations determined by

$$a + b = c \qquad \text{if, for all } n \in \mathbb{Z}_{>0}, \quad (a_{\le n} + b_{\le n})_{\le n-1} = c_{\le n-1} \text{ in } \mathbb{Q},$$

and

$$ab = c \qquad \text{if, for all } n \in \mathbb{Z}_{>0}, \quad (a_{\le n} b_{\le n})_{\le n-1} = c_{\le n-1} \text{ in } \mathbb{Q}.$$

An **irrational number** is a real number that is not a rational number.

Theorem 1.1. $\mathbb{Q} = \{\text{decimal expansions that repeat}\}$

Theorem 1.2. *Irrational numbers exist.*

The **complex numbers** is the set

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

with operations given by

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2) i \qquad \text{and}$$

$$(a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i.$$

Theorem 1.3. (The fundamental theorem of algebra) *If $p_0, p_1, \ldots, p_d \in \mathbb{C}$ with $p_d \neq 0$ then there are $\lambda_1, \ldots, \lambda_d \in \mathbb{C}$ such that*

$$p_0 + p_1 x + p_2 x^2 + \cdots + p_d x^d = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_d).$$

The **algebraic numbers** is the set

$$\overline{\mathbb{Q}} = \{z \in \mathbb{C} \mid \text{there exists } p(x) \in \mathbb{Q}[x],\ p(x) \neq 0, \text{ with } p(z) = 0\}.$$

A **transcendental number** is a complex number that is not an algebraic number.

References

[BouA] Bourbaki, N. *Algebra*, Springer, New York, ????

[BouC] Bourbaki, N. *Commutative Algebra*, Springer, New York, ????

[BouF] Bourbaki, N. *Functions of a Real Variable*, Springer, New York, ????

[BouI] Bourbaki, N. *Integration*, Springer, New York, ????

[BouS] Bourbaki, N. *Theory of Sets*, Springer, New York, ????

[BouT] Bourbaki, N. *General Topology*, Springer, New York, ????

[BouV] Bourbaki, N. *Topological Vector Spaces*, Springer, New York, ????

[Ru] Rudin, W. *Principles of Mathematical Analysis*, McGraw-Hill, New York, ????

[DS] Dunford and Schwartz, *??*, Wiley, New York, ????