Math 521: Lecture 10

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1 Cardinality

How big is a set?

Let S and T be sets. S and T have the **same cardinality**, $\mathrm{Card}(S) = \mathrm{Card}(T)$, if there is a bijective map from S to T.

Notation: Let S be a set. Then

$$\mathrm{Card}(S) = \begin{cases} 0, & \text{if } S = \emptyset, \\ n, & \text{if } \mathrm{Card}(S) = \mathrm{Card}\big(\{1, 2, \ldots, b\}\big) \\ \infty & \text{otherwise.} \end{cases}$$

Note: Even if $\mathrm{Card}(S) = \infty$ and $\mathrm{Card}(T) = \infty$, one may have that $\mathrm{Card}(S) \neq \mathrm{Card}(T)$.

A set S is **finite** if $\mathrm{Card}(S) \neq \infty$.

A set S is **infinite** if S is not finite.

A set S is **countable** if either S is finite or if $\mathrm{Card}(S) = \mathrm{Card}(\mathbb{Z}_{>0})$.

A set S is **uncountable** if S is not countable.

Let X be a topological space. A **perfect** set is a subset E of X such that

(a) E is closed,

(b) if $x \in E$ the x is a limit point of E.

The **Cantor set** is

$$C = \left\{ x \in [0, 1] \mid x \notin \left[\tfrac{2i-1}{3^k}, \tfrac{2i}{3^k}\right] \text{ for } k \in \mathbb{Z}_{>0} \text{ and } i \in \mathbb{Z}_{>0} \text{ with } 2i < 3^k \right\}$$

Theorem 1.1.

$$\mathrm{Card}(\mathbb{Z}_{>0}) = \mathrm{Card}(\mathbb{Z}_{\geq 0}) = \mathrm{Card}(\mathbb{Z}) = \mathrm{Card}(\mathbb{Q}) \neq \mathrm{Card}(\mathbb{R}) = \mathrm{Card}(\mathbb{C}).$$

HW: Show that subsets of countable sets are countable.

HW: Show that countable unions of countable sets are countable.

HW: Show that if S is countable and $n \in \mathbb{Z}_{>0}$ then S^n is countable.

HW: Show that $2^{\mathbb{Z}_{>0}}$ is uncountable.

HW; Show that $[a, b]$ is uncountable.

HW: Show that $\mathbb{R}$ is uncountable.

HW: Show that the Cantor set is uncountable.

HW: Show that every perfect subset of $\mathbb{R}^k$ is uncountable.

HW: Show that the Cantor set is perfect.