Math 521: Lecture 14

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1 Topology

A **topological space** is a set X with a specification of the **open** subsets of X where it is required that

(a) $\emptyset$ is open and X is open,

(b) Unions of open sets are open,

(c) Finite intersections of open sets are open.

In other words, a **topology** on X is a set $\mathcal{T}$ of subsets of X such that

(a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,

(b) If $U_i \in \mathcal{T}$ the $\bigcup_i U_i \in \mathcal{T}$,

(c) If $U_1, U_2, \cdots U_n$ is a finite collection of elements of $\mathcal{T}$ then $\bigcap_i U_i \in \mathcal{T}$.

A **topological space** is a set X with a topology $\mathcal{T}$ on X.

Let $\mathcal{T}$ be a topology on X. An **open set** is a set in $\mathcal{T}$.

A **closed set** is a subset E of X such that the complement E^c of E is open.

Let X be a topological space and let $x \in X$. A **neighborhood** of x is an open subset U of X such that $x \in U$.

Let X be a topological space. A **subspace** of X is a subset Y of X with the topology given by making the open sets be the sets

$$\iota^{-1}(V) \qquad \text{such that } V \text{ is an open subset of } X,$$

where $\iota \colon Y \to X$ is the inclusion.

2 Examples

Let X be a set. The **discrete topology** on X is the topology such that every subset of X is open.

A **metric space** is a set X with a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$ such that

(a) If $x \in X$ then $d(x, x) = 0$,

1. (b) If $x, y \in X$ and $d(x, y) = 0$ then $x = y$,

(c) If $x, y, z \in X$ then $d(x, z) \leq d(x, y) + d(y, z)$.

Let X be a metric space. Let $x \in X$ and let $\varepsilon \in \mathbb{R}_{>0}$. The **ball** of radius ε at x is the set

$$B_\varepsilon(x) = \{p \in X \mid d(x, p) \leq \varepsilon\}.$$

Let X be a metric space. The **metric space topology** on X is the topology generated by the sets

$$B_\varepsilon(x), \qquad \text{for } x \in X \text{ and } \varepsilon \in \mathbb{R}_{>0}.$$

3 Continuous functions

Continuous functions are for comparing topological spaces.

Let X and Y be topological spaces. A function $f\colon X \to Y$ is **continuous** if it satisfies the condition

$$\text{if } V \text{ is an open subset of } Y \text{ then } f^{-1}(V) \text{ is an open subset of } X.$$

Let X and Y be topological spaces. An **isomorphism** or **homeomorphism** is a continuous function $f\colon X \to Y$ such that the inverse function $f^{-1}\colon Y \to X$ exists and is continuous.

4 Interiors and Closures

Let $E \subset X$. The **interior** of E is the subset E° of E such that

(a) E° is open in X,

(b) If U is an open subset of E then $U \subseteq E^\circ$.

Let $E \subseteq X$. The **closure** $\bar{E}$ of E is the subset $\overline{E}$ of X such that

(a) $\overline{E}$ is closed,

(b) If V is a closed subset of X and $V \supseteq E$ then $V \supseteq \overline{E}$.

Let $E \subset X$. The set E is **dense** in X if $\overline{E} = X$.

5 Interior points and limit points

A **limit point** of E is a point $p \in \bar{E} \backslash E^0$.

Let $E \subseteq X$. An **interior point** of E is an element $p \in E$ such that there exists a neighborhood U of p with $U \subseteq E$.

6 Compact sets

Let $K \subseteq X$. A **sequence** of points of K is a subset $(k_1, k_2, k_3, \ldots)$ of elements K indexed by the elements of $\mathbb{Z}_{>0}$.

A **compact** subset of X is a subset K of X such that any open cover of K has a finite subcover. In other words, if $\{U_\alpha\}$ is a collection of open subsets of X such that $K \subseteq \cup_\alpha U_\alpha$ then there is a finite subset $\{U_1, \ldots, U_n\}$ of $\{U_\alpha\}$ such that $K \subseteq U_1 \cup \cdots \cup U_n$.

Let $K \subseteq X$. A **sequence** of points of K is a subset $(k_1, k_2, k_3, \ldots)$ of elements K indexed by the elements of $\mathbb{Z}_{>0}$.

Theorem 6.1. *Let X be a topological space and let $K \subseteq X$. Then K is compact if and only if every sequence $(k_1, k_2, \ldots,)$ of points of K has a limit point in K.*