Math 521: Lecture 17

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1 Interiors and closures

Let X be a topological space and let $x \in X$. A **neighborhood** of x is a subset N of X such that there exists an open subset U of X with $x \in U$ and $U \subseteq N$.

Let X be a topological space and let $E \subset X$. A **neighborhood** of E is a subset N of X such that there exists an open subset U of X with $E \subseteq U \subseteq N$.

Let X be a topological space and let $E \subset X$. The **interior** of E is the subset E° of E such that

(a) E° is open in X,

(b) If U is an open subset of E then $U \subseteq E^\circ$.

Let X be a topological space and let $E \subseteq X$. The **closure** $\bar{E}$ of E is the subset $\overline{E}$ of X such that

(a) $\overline{E}$ is closed,

(b) If V is a closed subset of X and $V \supseteq E$ then $V \supseteq \overline{E}$.

Let X be a topological space. Let $E \subseteq X$. An **interior point** of E is a point $x \in X$ such that there exists a neighborhood N_x of x with $N_x \subseteq E$.

Let X be a topological space. Let $E \subseteq X$. A **close point** to E is a point $x \in X$ such that If N_x is a neighborhood of of x then N_x contains a point of E.

Theorem 1.1. *Let X be a topological space. Let $E \subseteq X$.*

(a) The interior of E is the set of interior points of E.

(b) The closure of E is the set of close points of E.

2 Hausdorff spaces

A **Hausdorff space** is a topological space X such that if $x, y \in Y$ and $x \neq y$ then there exist a neighborhood N_x of x and a neighborhood N_y of y such that $N_x \cup N_y = \emptyset$.

Theorem 2.1. *Let X be a topological space. Show that the following are equivalent:*

(a) *Any two distinct points of X have disjoint neighborhoods.*

(b) *The intersection of the closed neighborhoods of any point of X consist of that point alone.*

(c) *The diagonal of the product space $X \times X$ is a closed set.*

(d) *For every set I, the diagonal of the product space $Y = X^I$ is closed in Y.*

(e) *No filter on X has more than one limit point.*

(f) *If a filter $\mathcal{F}$ on X converges to x then x is the only cluster point of x.*

3 Limit points and cluster points

Theorem 3.1. *Let X be a topological space and let $(x_1, x_2, \ldots)$ be a sequence in X. Then*

(a) *y is a limit point of $(x_1, x_2, \ldots)$ if and only if, if N_y is a neighborhood of y then there exists $n_0 \in \mathbb{Z}_{>0}$ such that $x_n \in N_x$ for all $n \in \mathbb{Z}_{\geq 0}$, $n \geq n_0$.*

(b) *y is a cluster point of $(x_1, x_2, \ldots)$ if and only if, if N_y is a neighborhood of y and $n_0 \in \mathbb{Z}_{>0}$ then there exists $n \in \mathbb{Z}_{>0}$ with $n \geq n_0$ such that $x_n \in N_y$.*

4 Compact sets

Let X be a set. A filter $\mathcal{F}$ on X is **convergent** if it has a limit point.

Theorem 4.1. *Let X be a topological space. The following are equivalent.*

(a) *Every filter on X has at least one cluster point.*

(b) *Every ultrafilter on X is convergent.*

(c) *Every family of closed subsets of X whose intersection is empty contains a finite subfamily whose intersection is empty.*

(d) *Every open cover of X contains a finite subcover.*