Math 521: Lecture 18

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1 Interiors and closures

Let X be a topological space and let $x \in X$. A **neighborhood** of x is a subset N of X such that there exists an open subset U of X with $x \in U$ and $U \subseteq N$.

Let X be a topological space and let $E \subset X$. A **neighborhood** of E is a subset N of X such that there exists an open subset U of X with $E \subseteq U \subseteq N$.

Let X be a topological space and let $E \subset X$. The **interior** of E is the subset E° of E such that

(a) E° is open in X,

(b) If U is an open subset of E then $U \subseteq E^\circ$.

Let X be a topological space and let $E \subseteq X$. The **closure** $\bar{E}$ of E is the subset $\overline{E}$ of X such that

(a) $\overline{E}$ is closed,

(b) If V is a closed subset of X and $V \supseteq E$ then $V \supseteq \overline{E}$.

Let X be a topological space. Let $E \subseteq X$. An **interior point** of E is a point $x \in X$ such that there exists a neighborhood N_x of x with $N_x \subseteq E$.

Let X be a topological space. Let $E \subseteq X$. A **close point** to E is a point $x \in X$ such that If N_x is a neighborhood of of x then N_x contains a point of E.

Theorem 1.1. *Let X be a topological space. Let $E \subseteq X$.*

(a) The interior of E is the set of interior points of E.

(b) The closure of E is the set of close points of E.

2 Hausdorff spaces

A **Hausdorff space** is a topological space X such that if $x, y \in Y$ and $x \neq y$ then there exist a neighborhood N_x of x and a neighborhood N_y of y such that $N_x \cup N_y = \emptyset$.

Theorem 2.1. *Let X be a topological space. Show that the following are equivalent:*

(a) *Any two distinct points of X have disjoint neighborhoods.*

(b) *The intersection of the closed neighborhoods of any point of X consist of that point alone.*

(c) *The diagonal of the product space $X \times X$ is a closed set.*

(d) *For every set I, the diagonal of the product space $Y = X^I$ is closed in Y.*

(e) *No filter on X has more than one limit point.*

(f) *If a filter $\mathcal{F}$ on X converges to x then x is the only cluster point of x.*

3 Limit points and cluster points

Theorem 3.1. *Let X be a topological space and let $(x_1, x_2, \ldots)$ be a sequence in X. Then*

(a) *y is a limit point of $(x_1, x_2, \ldots)$ if and only if, if N_y is a neighborhood of y then there exists $n_0 \in \mathbb{Z}_{>0}$ such that $x_n \in N_x$ for all $n \in \mathbb{Z}_{\geq 0}$, $n \geq n_0$.*

(b) *y is a cluster point of $(x_1, x_2, \ldots)$ if and only if, if N_y is a neighborhood of y and $n_0 \in \mathbb{Z}_{>0}$ then there exists $n \in \mathbb{Z}_{>0}$ with $n \geq n_0$ such that $x_n \in N_y$.*

4 Compact sets

Let X be a set. A filter $\mathcal{F}$ on X is **convergent** if it has a limit point.

Theorem 4.1. *Let X be a topological space. The following are equivalent.*

(a) *Every filter on X has at least one cluster point.*

(b) *Every ultrafilter on X is convergent.*

(c) *Every family of closed subsets of X whose intersection is empty contains a finite subfamily whose intersection is empty.*

(d) *Every open cover of X contains a finite subcover.*

Proof. (b) $\Rightarrow$ (a): Let $\mathcal{F}$ be a filter and let $\overline{\mathcal{F}}$ be an ultrafilter containing $\mathcal{F}$. Let x be a limit point of $\overline{\mathcal{F}}$. Then x is a limit point of $\mathcal{F}$.

(a) $\Rightarrow$ (b): Let $\overline{\mathcal{F}}$ be an ultrafilter. Let x be a cluster point of $\overline{\mathcal{F}}$. Then x is a limit point of $\overline{\mathcal{F}}$. So $\overline{\mathcal{F}}$ is convergent.

(a) $\Rightarrow$ (c): Let $\mathcal{C}$ be a closed family with empty intersection. If every finite subfamily has empty intersection then $\mathcal{C}$ generates a filter $\mathcal{F}$. Let x be a cluster point of $\mathcal{F}$. Then $x \in C$ for every set C in $\mathcal{C}$. This is a contradiction. So there exists a finite subfamily that does not have empty intersection.

(c) $\Rightarrow$ (a): If there exists a filter $\mathcal{F}$ without a cluster point then $\mathcal{C} = \{\overline{F} \mid f \in \mathcal{F}\}$ is a family of closed sets contradicting (c).

(c) $\Leftrightarrow$ (d) by taking complements.

□

Theorem 4.2. *Let X be a metric space and let E be a subset of X. The set E is compact if and only if every infinite subset of E has a limit point in E.*

Proof. $\Leftarrow$: Let K be a compact set and let E be a infinite subset of K. If there is no limit point of E in K then, for each $p \in K$ there is a neighborhood N_p which contains no other element of E. Then the open cover

$$\mathcal{N} = \{N_p \mid p \in K\},$$

of K has no finite subcover.

$\Rightarrow$: Let S be a infinite subset of E. The metric space E has a countable base. So every open cover of E has a countable subcover $\mathcal{C} = \{C_1, C_2, \ldots\}$. If $\mathcal{C}$ does not have a finite subcover then, for each n, $(C_{\le n})^c \neq \emptyset$ but $\bigcap_n C^c_{\le n} = \emptyset$. Let S be a set which contains a point from each $C^c_{\le n}$. Then S has a limit point. But this is a contradiction. □

Theorem 4.3. *Let X be a Hausdorff topological space and let K be a compact subset of X. Then K is closed.*

Proof. Let $x \in \overline{K}$. The neighborhood filter $\mathcal{B}(x)$ of x induces a filter $\mathcal{B}_K$ on K which has a cluster point $y \in K$. Since $\mathcal{B}(x)$ is coarser than $\mathcal{B}_K$ (considered as a filter base on X) the point y is a cluster point of $\mathcal{B}(x)$. So $y = x$ since X is Hausdorff.

The proof in baby Rudin: We will show that K^c is open. Let $p \in K^c$. Let $\mathcal{N}$ be the open cover of K given by

$$\mathcal{N} = \{N_q \mid q \in K\}, \quad \text{where} \quad N_q = B_{\frac{1}{2}d(p,q)}(q) \mid q \in K\}.$$

Let $\{N_{q_1}, \ldots, N_{q_\ell}\}$ be a finite subcover of K. Then

$$M = M_{q_1} \cap \cdots \cap M_{q_\ell}, \quad \text{where} \quad M_q = B_{\frac{1}{2}d(p,q)}(p),$$

is an open set such that $p \in M \subseteq K^c$. So p is an interior point of K^c. So K is open. □

Theorem 4.4. *Let X be a metric space and let E be a compact subset of X. Then E is closed and bounded.*

Proof. Since a metric space is Hausdorff, E is closed. If E is not bounded then there is an infinite sequence in E that does not have a limit point. □

Theorem 4.5. *(a) A k-cell is compact.*

(b) Let E be a subset of $\mathbb{R}^k$. If E is closed and bounded then E is compact.

Proof. If E is closed and bounded then E is a closed subset of a k-cell. Since closed subsets of compact sets are compact E is compact. □