Math 521: Lecture 4

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1 Relations

A **relation** on a set S is a subset of $S \times S$. Write $s_1 \sim s_2$ if the pair (s_1, s_2) is in the relation.

Let S be a set and let $\sim$ be a relation on S. The relation $\sim$ is **reflexive** if it satisfies the condition

$$\text{If } s \in S \quad \text{then } s \sim s.$$

The relation $\sim$ is **symmetric** if it satifies the condition

$$\text{If } s_1, s_2 \in S \text{ and } s_1 \sim s_2 \text{ then } s_2 \sim s_1.$$

The relation $\sim$ is **transitive** if it satisfies the condition

$$\text{If } s_1, s_2, s_3 \in S \text{ and } s_1 \sim s_2 \text{ and } s_2 \sim s_3 \text{ then } s_1 \sim s_3.$$

An **equivalence relation** on a set S is a relation on S that is reflexive, symmetric and transitive.

Example. Let S be the set $\{1, 2, 6\}$. Then

(a) $R_1\{(1,1), (2,6), (6,1)\}$ is a relation on S.

(b) R_1 is not reflexive, not symmetric, and not transitive.

(c) $R_2 = \{(1,1), (2,6), (6,1), (2,1)\}$ is a relation on S.

(d) R_2 is transitive but not symmetric and not reflexive.

Let S be a set and let $\sim$ be an equivalence relation on S. The **equivalence class** of an element $s \in S$ is the set

$$[s] = \{t \in S \mid t \sim s\}.$$

Let S be a set. A **cover** of S is a collection of subsets S_α such that

$$\text{If } s \in S \text{ then } s \in S_\alpha \text{ for some } S_\alpha.$$

Let S be a set. A **partition** of S is a collection of subsets S_α such that

(a) If $s \in S$ then $s \in S_\alpha$ for some S_α.

(b) If $S_\alpha \cap S_\beta \neq \emptyset$ then $S_\alpha = S_\beta$.

Proposition 1. *(a) Let S be a set and let $\sim$ be an equivalence relation on S. The set of equivalence classes of the relation $\sim$ is a partition of S.*

(b) Let S be a set and let $\{S_\alpha\}$ be a partition of S. Then the relation defined by

$$s \sim t \text{ if } s \text{ and } t \text{ are in the same } S_\alpha$$

is an equivalence relation on S.

Proposition ??? shows that the concepts of an equivalence relation on S and of a partition of S are essentially the same. Each equivalence relation on S determines a partition on S and vice versa.

Example. Let $S = \{1, 2, 3, \ldots, 10\}$. Let $\sim$ be the equivalence relation determined by

$$1 \sim 5, \quad 2 \sim 3, \quad 9 \sim 10, \quad 1 \sim 7, \quad 5 \sim 8, \quad 10 \sim 4.$$

Since we are requiring that $\sim$ is an equivalence relation, we are assuming that we have all the other relations we need so that $\sim$ is reflexive, symmetric, and transitive:

$$1 \sim 1,\ 2 \sim 2,\ \ldots,\ 10 \sim 10,$$
$$5 \sim 7,\ 7 \sim 8,\ 7 \sim 5,\ 5 \sim 1,\ \ldots.$$

Then the equivalence classes are given by

$$\begin{aligned} [1] = [5] = [7] = [8] &= \{1, 5, 7, 8\} \\ [2] = [3] &= \{2, 3\} \\ [6] &= \{6\} \\ [4] = [9] = [10] &= \{4, 9, 10\}, \end{aligned}$$

and the sets

$$S_1 = \{1, 5, 7, 8\},\ S_2 = \{2, 3\},\ S_3 = \{6\},\ \text{and } S_4 = \{4, 9, 10\}$$

form a partition of S.