Math 521: Lecture 4

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1 Polynomials

Let $\mathbb{F}$ be a field. If $a_0, a_1, a_2, \ldots \in \mathbb{F}$ use the notation

$$a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i \in \mathbb{Z}_{\geq 0}} a_i x^i.$$

The **polynomial ring** is the set

$$\mathbb{F}[x] = \left\{ \sum_{i \in \mathbb{Z}_{\geq 0}} \mid a_i \in \mathbb{F} \text{ and all but a finite number of the } a_i \text{ are equal to } 0 \right\},$$

with operations given by

$$\left(\sum_{i \in \mathbb{Z}_{\geq 0}} a_i x^i \right) + \left(\sum_{i \in \mathbb{Z}_{\geq 0}} b_i x^i \right) = \left(\sum_{i \in \mathbb{Z}_{\geq 0}} (a_i + b_i) x^i \right)$$

and

$$\left(\sum_{i \in \mathbb{Z}_{\geq 0}} a_i x^i \right) \left(\sum_{j \in \mathbb{Z}_{\geq 0}} b_j x^j \right) = \left(\sum_{k \in \mathbb{Z}_{\geq 0}} c_k x^i \right), \qquad \text{where} \quad c_k = \sum_{i+j=k} a_i b_j.$$

Let $a \in \mathbb{F}$. The **evaluation homomorphism** is

$$\begin{array}{rccc} \mathrm{ev}_a\colon & \mathbb{F}[x] & \longrightarrow & \mathbb{F} \\ & p(x) & \longmapsto & p(a) \end{array}$$

where

$$p(a) = p_0 + p_1 a + p_2 a^2 + \cdots \qquad \text{if} \qquad p(x) = p_0 + p_1 x + p_2 x^2 + \cdots.$$

Let $p(x) \in \mathbb{F}[x]$, $p(x) = p_0 + p_1 x + p_2 x^2 + \cdots$. The **degree** $\deg(p(x))$ of $p(x)$ is the maximal nonnegative integer d such that $p_d \neq 0$.

The **field of rational functions** in x is the set

$$\mathbb{F}(x) = \left\{ \frac{a(x)}{b(x)} \mid a(x), b(x) \in \mathbb{F}[x], \quad b(x) \neq 0 \right\},$$

with

$$\frac{a(x)}{b(x)} = \frac{c(x)}{d(x)}, \qquad \text{if} \quad a(x)d(x) = b(x)c(x),$$

and with operations given by

$$\frac{a(x)}{b(x)} + \frac{c(x)}{d(x)} = \frac{a(x)d(x) + b(x)c(x)}{b(x)d(x)} \qquad \text{and} \qquad \frac{a(x)}{b(x)} \cdot \frac{c(x)}{d(x)} = \frac{a(x)c(x)}{b(x)d(x)}.$$

The **ring of formal power series** in x is

$$\mathbb{F}[x] = \left\{ \sum_{i \in \mathbb{Z}_{\geq 0}} a_i x^i \mid a_i \in \mathbb{F} \right\},$$

with operations given by

$$\left(\sum_{i \in \mathbb{Z}_{\geq 0}} a_i x^i \right) + \left(\sum_{i \in \mathbb{Z}_{\geq 0}} b_i x^i \right) = \left(\sum_{i \in \mathbb{Z}_{\geq 0}} (a_i + b_i) x^i \right)$$

and

$$\left(\sum_{i \in \mathbb{Z}_{\geq 0}} a_i x^i \right) \left(\sum_{j \in \mathbb{Z}_{\geq 0}} b_j x^j \right) = \left(\sum_{k \in \mathbb{Z}_{\geq 0}} c_k x^i \right), \qquad \text{where} \quad c_k = \sum_{i+j=k} a_i b_j.$$

Examples.

(1) $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots.$

(2) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots = \sum_{i \in \mathbb{Z}_{\geq 0}} \dfrac{x^i}{i!}.$

(3) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots = \sum_{i \in \mathbb{Z}_{\geq 0}} (-1)^i \dfrac{x^{(2i+1)}}{(2i+1)!}.$

(4) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots = \sum_{i \in \mathbb{Z}_{\geq 0}} (-1)^i \dfrac{x^{2i}}{(2i)!}.$

(5) $\ln(1-x) = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots = \sum_{i \in \mathbb{Z}_{>0}} \dfrac{x^i}{i}.$

The **Laurent polynomials** is the set

$$\mathbb{F}(x) = \left\{ \frac{a(x)}{b(x)} \mid a(x), b(x) \in \mathbb{F}[[x]], \quad b(x) \neq 0 \right\},$$

with

$$\frac{a(x)}{b(x)} = \frac{c(x)}{d(x)}, \qquad \text{if} \quad a(x)d(x) = b(x)c(x),$$

and with operations given by

$$\frac{a(x)}{b(x)} + \frac{c(x)}{d(x)} = \frac{a(x)d(x) + b(x)c(x)}{b(x)d(x)} \qquad \text{and} \qquad \frac{a(x)}{b(x)} \cdot \frac{c(x)}{d(x)} = \frac{a(x)c(x)}{b(x)d(x)}.$$

Proposition 1. *(a)* $\mathbb{F}[x]$ *is an integral domain.*

(b) $\mathbb{F}[[x]]$ *is an integral domain.*

Proposition 2. *(a) The invertible elements of* $\mathbb{F}[x]$ *are the invertible elements of* $\mathbb{F}$*.*

(b) The invertible elements of $\mathbb{F}[[x]]$ *are* $a_0 + a_1x + a_2x^2 + \cdots \in \mathbb{F}[[x]]$ *with* a_0 *invertible in* $\mathbb{F}$*.*

Corollary 1. $\mathbb{F}((x)) = \{x^k p(x) \mid p(x) \in \mathbb{F}[[x]], p_0 \neq 0\}$.

Let $p(x) \in \mathbb{F}((x))$. The **order** $\nu(p(x))$ of $p(x) = \sum_{\ell \in \mathbb{Z}} p_\ell x^\ell$ is the minimal integer ℓ such that $p_\ell \neq 0$.

The order function $\nu \colon \mathbb{F}((x)) \to \mathbb{Z}$ is a *normalized discrete valuation* (see [BouC] Ch. VI §3 no. 6 Def. 3).