Math 521: Lecture 8

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1 The exponential function

Let $k \in \mathbb{Z}_{\geq 0}$ define k **factorial** by

$$0! = 0 \qquad \text{and} \qquad k! = k \cdot (k-1) \cdots 3 \cdot 2 \cdot 1, \quad \text{if } k \in \mathbb{Z}_{\geq 0}.$$

Let $n, k \in \mathbb{Z}_{\geq 0}$ with $k \leq n$. Define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Theorem 1.1. *Let $n, k \in \mathbb{Z}_{\geq 0}$ with $k \leq n$.*

(a) Let S be a set of cardinality n. Then $\binom{n}{k}$ is the number of subsets of S of cardinality k.

(b) $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in $(x+y)^n$.

(c) $\binom{n}{n} = 1$, $\binom{n}{0} = 1$ and, if $1 \leq k \leq n-1$ then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

The **exponential function** is the element e^x of $\mathbb{Q}[[x]]$ given by

$$e^x = \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{x^k}{k!}.$$

Theorem 1.2. *As elements of $\mathbb{Q}[[x, y]]$,*

$$e^x e^y = e^{(x+y)}.$$

Define

$$\ln(1+x) = \sum_{k \in \mathbb{Z}_{>0}} (-1)^{k-1} \frac{x^k}{k}.$$

Theorem 1.3. *Let*

$$G = \{p(x) \in \mathbb{F}[[x]] \mid p(0) = 1\} \qquad \textit{and} \qquad \mathfrak{g} = \{p(x) \in \mathbb{F}[[x]] \mid p(0) = 0\}.$$

(a) $\ln(1 + (e^x - 1)) = e^{\ln(1+x)} - 1 = x.$

(b) G is an abelian group under multiplication, $\mathfrak{g}$ is a commutative group under addition and

$$\begin{array}{ccc} G & \longrightarrow & \mathfrak{g} \\ p & \longmapsto & e^p - 1 \end{array}$$

is an isomorphism of groups.