Math 521: Lecture 9

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1 Derivations

Let $\mathbb{F}$ be a field. A **vector space** over $\mathbb{F}$ is an abelian group V with a map

$$\begin{array}{ccc} \mathbb{F}\times V & \longrightarrow & V \\ (c,v) & \longmapsto & cv \end{array}$$

such that

(a) If $c_1, c_2 \in \mathbb{F}$ and $v \in V$ then $(c_1+c_2)v = c_1v + c_2v$,

(b) If $c \in \mathbb{F}$ and $v_1, v_2 \in V$ then $c(v_1+v_2) = cv_1 + cv_2$,

(c) If $c_1, c_2 \in \mathbb{F}$ and $v \in V$ then $c_1(c_2v) = (c_1c_2)v$,

(d) If $v \in V$ then $1 \cdot v = v$.

Let $\mathbb{F}$ be a field. Let V, W be vector spaces over $\mathbb{F}$. An **$\mathbb{F}$-linear map** from V to W is a function $\varphi\colon V \to W$ such that

(a) φ is a group homomorphism,

(b) If $c \in \mathbb{F}$ and $v \in V$ then $\varphi(cv) = c\varphi(v)$.

Let $\mathbb{F}$ be a field. An **algebra** is a vector space A over $\mathbb{F}$ with an operation

$$\begin{array}{ccc} A\times A & \longrightarrow & A \\ (a_1,a_2) & \longmapsto & a_1a_2 \end{array}$$

such that A is a ring and the scalar multiplication is the composition of the map

$$\begin{array}{ccc} \mathbb{F} & \longrightarrow & A \\ \xi & \longmapsto & \xi\cdot 1 \end{array}$$

and the multiplication in A.

Let $\mathbb{F}$ be a field. Let A be an $\mathbb{F}$-algebra. A **derivation** of A is an $\mathbb{F}$-linear map $d\colon A \to A$ such that

$$\text{if} \quad a_1, a_2 \in A \quad \text{then} \quad d(a_1a_2) = a_1d(a_2) + d(a_1)a_2.$$

Theorem 1.1. *(a) There is a unique derivation* $\frac{d}{dx}$ *of* $\mathbb{F}[x]$ *such that* $\frac{dx}{dx} = 1$.

(b) If $p \in \mathbb{F}[x]$ *then*

$$\frac{dp}{dx} = (\textit{coefficient of } y \textit{ in } p(x+y)).$$

(c) If $p \in \mathbb{F}[x]$ *then*

$$p = \sum_{k \in \mathbb{Z}_{\geq 0}} \left(\left(\frac{d}{dx} \right)^k p \right)(0) x^k.$$

(d) There is a unique extension of $\frac{d}{dx}$ *to a derivation of* $\mathbb{F}(x)$.

(e) There is a unique extension of $\frac{d}{dx}$ *to a derivation of* $\mathbb{F}[[x]]$.

(f) There is a unique extension of $\frac{d}{dx}$ *to a derivation of* $\mathbb{F}((x))$.

(g) If $p \in \mathbb{F}[[x]]$ *then*

$$\frac{dp}{dx} = (\textit{coefficient of } y \textit{ in } p(x+y)).$$

(h) If $p \in \mathbb{F}[[x]]$ *then*

$$p = \sum_{k \in \mathbb{Z}_{\geq 0}} \left(\left(\frac{d}{dx} \right)^k p \right)(0) x^k.$$