Chapter 0. SETS AND FUNCTIONS

The basic building blocks of mathematics are sets and functions. Functions allow us to compare sets.

§1T. Sets

(0.1.1) Definition.

- A **set** is a collection of objects which are called **elements**. We write $s \in S$ if s is an element of a set S.
- The **emptyset**, $\emptyset$, is the set with no elements.
- A **subset** T of a set S is a set T such that if $t \in T$ then $t \in S$. We write $T \subseteq S$.
- Two sets S and T are **equal** if $S \subseteq T$ and $T \subseteq S$. We write $T = S$.
- Let S and T be sets. The **union** of S and T is the set $S \cup T$ of all u such that $u \in S$ or $u \in T$.

$$S \cup T = \{u \mid u \in S \text{ or } u \in T\}.$$

- Let S and T be sets. The **intersection** of S and T is the set $S \cap T$ of all u such that $u \in S$ and $u \in T$.

$$S \cap T = \{u \mid u \in S \text{ and } u \in T\}.$$

- Let S and T be sets. S and T are **disjoint** if $S \cap T = \emptyset$.
- Let S and T be sets. S is a **proper subset** of T if $S \subseteq T$ and $S \neq T$. We write $S \subsetneq T$.
- The **product** of two sets S and T is the set of all ordered pairs (s,t) where $s \in S$ and $t \in T$,

$$S \times T = \{(s,t) \mid s \in S, t \in T\}.$$

  More generally, given sets $S_1, \ldots, S_n$, the **product** $\prod_i S_i$ is the set of all tuples $(s_1, \ldots, s_n)$ such that $s_i \in S_i$.
- The elements of a set S are **indexed** by the elements of a set I if each element of S is labeled by a unique element of I. If $i \in I$, s_i denotes the corresponding element of S.

We will use the following notations:

$$\begin{aligned}
\mathbb{Z} &= \{\ldots, -2, -1, 0, 1, 2, \ldots\} \quad \text{is the set of integers.}\\
\mathbb{N} &= \{0, 1, 2, \ldots\} \quad \text{is the set of nonnegative integers.}\\
\mathbb{P} &= \{1, 2, \ldots\} \quad \text{is the set of positive integers.}\\
[1,n] &= \{1, 2, \ldots, n\} \quad \text{for each } n \in \mathbb{P}.\\
\mathbb{Q} &= \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{P}\} \quad \text{is the set of rational numbers.}\\
\mathbb{R} & \quad \text{is the set of real numbers.}\\
\mathbb{C} & \quad \text{is the set of complex numbers.}
\end{aligned}$$

Example. Let S, T, U, and V be the sets $S = \{1,2\}$, $U = \{1,2\}$, $T = \{1,2,3\}$, and $V = \{2,3\}$. Then

a) $S \subseteq U \subseteq T$.
b) $U \not\subseteq V$.
c) $U \cup V = T$.
d) $U \cap V = \{2\}$.
e) $S \times T = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$.

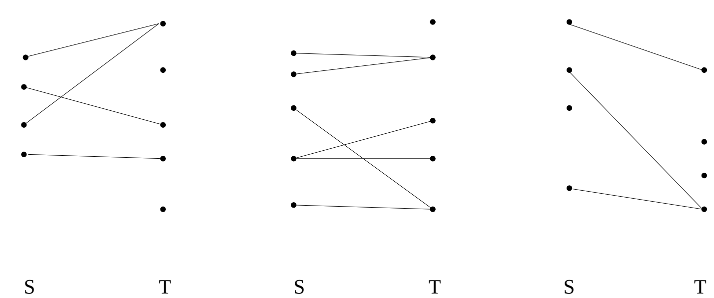
HW: Show that the emptyset is a subset of every set.

§2T. Functions

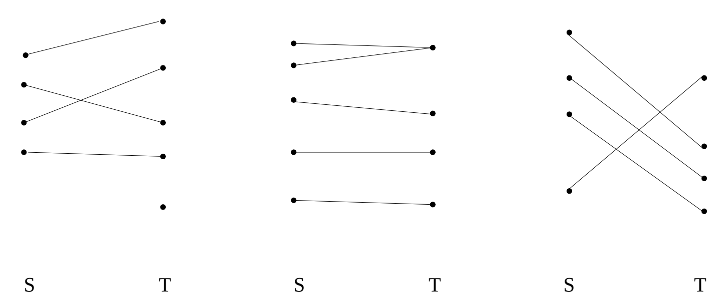
(2.2.1) Definition.

- Let S and T be sets. A **map** or **function** $f\colon S \to T$ is given by associating to each element $s \in S$ a unique element $f(s) \in T$.
$$\begin{array}{rccc} f\colon & S & \to & T \\ & s & \mapsto & f(s). \end{array}$$
- Often in mathematics one will try to define a function without being exactly sure if what has been defined really is a function. In order to check that a function is **well defined** one must check that
  a) For every $s \in S$, $f(s) \in T$.
  b) If $s_1 = s_2$ then $f(s_1) = f(s_2)$.
- Let S and T be sets. Two functions $f\colon S \to T$ and $g\colon S \to T$ are **equal** if
$$f(s) = g(s), \quad \text{for all} \quad s \in S.$$
We write $f = g$.
- Let S and T be sets and let $f\colon S \to T$ be a function. Let $R \subseteq S$. The **restriction** of f to R is the function $f|_R$ given by
$$\begin{array}{rccc} f|_R\colon & R & \to & T \\ & r & \mapsto & f(r). \end{array}$$
- A map $f\colon S \to T$ is **injective** or **one-to-one** if it satisfies
$$\text{If } s_1, s_2 \in S \text{ and } f(s_1) = f(s_2) \text{ then } s_1 = s_2.$$
- A map $f\colon S \to T$ is **surjective** or **onto** if for each element $t \in T$ there exists $s \in S$ such that $f(s) = t$.
- A map is **bijective** if it is both injective and surjective.

Examples. It is useful to visualize a function $f\colon S \to T$ as a graph with edges $\big(s, f(s)\big)$ connecting elements of $s \in S$ and $f(s) \in T$. With this idea in mind we have the following.



a) function b) not a function c) not a function



d) injective function e) surjective function f) bijective function

In these pictures we are viewing the elements of the left column as elements of the set S and the elements of the right column as the elements of a set T. In order to be a function the graph must have exactly one edge adjacent to each element of S. A function is injective if there is at most one edge adjacent to each point of T. A function is surjective if there is at least one edge adjacent to each point of T.

Composition of Functions

(2.2.2) Definition.

- Let $f\colon S \to T$ and $g\colon T \to U$ be functions. The **composition** of f and g is the function $g \circ f$ given by

$$\begin{array}{rccc} (g \circ f)\colon & S & \to & U \\ & s & \mapsto & g\big(f(s)\big). \end{array}$$

- Let S be a set. The **identity map** on a set S is the map given by

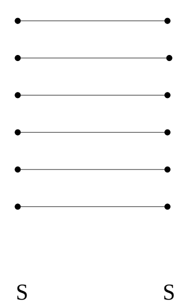
$$\begin{array}{rccc} \iota_S\colon & S & \to & S \\ & s & \mapsto & s. \end{array}$$

- Let $f\colon S \to T$ be a function. An **inverse function** to f is a function $f^{-1}\colon T \to S$ such that

$$f \circ f^{-1} = \iota_T \quad \text{and}$$
$$f^{-1} \circ f = \iota_S$$

  where ι_T and ι_S are the identity functions on T and S respectively.

If we visualize functions as graphs, the identity function ι_S looks something like



The function ι_S

In the pictures below, if the left graph is a pictorial representation of a function $f\colon S \to T$ then the inverse function to f, $f^{-1}\colon T \to S$, is represented by the graph on the right.

f $\qquad\qquad$ f^{-1}

(2.2.3) Proposition. *Let $f: S \to T$ be a function. An inverse function to f exists if and only if f is bijective.*

Pictorially, the graph, below left, represents a function $g: S \to T$ which is not bijective. The inverse function to g does not exist in this case; the graph of a possible candidate (below right) is not the graph of a function.

g not a function

Operations

(2.2.4) Definition.

- An **operation** on a set S is a map $\circ: S \times S \to S$. If $s_1, s_2 \in S$ we write $s_1 \circ s_2$ instead of $\circ\big((s_1, s_2)\big)$.
- An operation on a set S is **associative** if, for all $s_1, s_2, s_3 \in S$,

$$(s_1 \circ s_2) \circ s_3 = s_1 \circ (s_2 \circ s_3).$$

- An operation on a set S is **commutative** if, for all $s_1, s_2 \in S$,

$$s_1 \circ s_2 = s_2 \circ s_1.$$

Example. The map $+: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by

$$\begin{array}{rccc} +: & \mathbb{Z} \times \mathbb{Z} & \to & \mathbb{Z} \\ & (i, j) & \mapsto & i + j \end{array}$$

is an operation. This operation is both commutative and associative.

The map $-: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by

$$\begin{array}{rccc} -: & \mathbb{Z} \times \mathbb{Z} & \to & \mathbb{Z} \\ & (i, j) & \mapsto & i - j \end{array}$$

is an operation. This operation is noncommutative and nonassociative.

Relations

(2.2.5) Definition.

- A **relation** on a set S is a subset of $S \times S$. We write $s_1 \sim s_2$ if the pair (s_1, s_2) is in this subset.
- A relation is **reflexive** if, for each $s \in S$,

$$s \sim s.$$

- A relation is **symmetric** if
$$s_1 \sim s_2 \iff s_2 \sim s_1.$$
- A relation is **transitive** if
$$s_1 \sim s_2 \text{ and } s_2 \sim s_3 \Longrightarrow s_1 \sim s_3.$$
- An **equivalence relation** on a set S is a relation on S that is reflexive, symmetric and transitive.

Example. Let S be the set $\{1, 2, 6\}$. Then:

a) $R_1\{(1,1), (2,6), (6,1)\}$ is a relation on S.
b) R_1 is not reflexive, not symmetric, and not transitive.
c) $R_2 = \{(1,1), (2,6), (6,1), (2,1)\}$ is a relation on S.
d) R_2 is transitive but not symmetric and not reflexive.

(2.2.6) Definition.

- Let S be a set and let $\sim$ be an equivalence relation on S. The **equivalence class** of an element $s \in S$ is the set
$$[s] = \{t \in S \mid t \sim s\}.$$
- A **partition** of a set S is a collection of subsets S_α such that:
  a) If $s \in S$ then $s \in S_\alpha$ for some S_α.
  b) If $S_\alpha \cap S_\beta \neq \emptyset$ then $S_\alpha = S_\beta$.

(2.2.7) Proposition.

a) *Let S be a set and let $\sim$ be an equivalence relation on S. The set of equivalence classes of the relation $\sim$ is a partition of S.*
b) *Let S be a set and let $\{S_\alpha\}$ be a partition of S. Then the relation defined by*
$$s \sim t \text{ if } s \text{ and } t \text{ are in the same } S_\alpha$$
is an equivalence relation on S.

Proposition 2.2.7 shows that the concepts of an equivalence relation on S and of a partition of S are essentially the same. Each equivalence relation on S determines a partition on S and vice versa.

Example. Let $S = \{1, 2, 3, \ldots, 10\}$. Let $\sim$ be the equivalence relation determined by
$$\begin{array}{lll} 1 \sim 5, & 2 \sim 3, & 9 \sim 10, \\ 1 \sim 7, & 5 \sim 8, & 10 \sim 4. \end{array}$$
Since we are requiring that $\sim$ is an equivalence relation, we are assuming that we have all the other relations we need so that $\sim$ is reflexive, symmetric, and transitive:
$$1 \sim 1,\ 2 \sim 2,\ \ldots,\ 10 \sim 10,$$
$$5 \sim 7,\ 7 \sim 8,\ 7 \sim 5,\ 5 \sim 1,\ \ldots.$$
Then the equivalence classes are given by
$$\begin{aligned} [1] = [5] = [7] = [8] &= \{1, 5, 7, 8\} \\ [2] = [3] &= \{2, 3\} \\ [6] &= \{6\} \\ [4] = [9] = [10] &= \{4, 9, 10\}, \end{aligned}$$
and the sets
$$S_1 = \{1, 5, 7, 8\},\ S_2 = \{2, 3\},\ S_3 = \{6\},\ \text{and } S_4 = \{4, 9, 10\}$$
form a partition of S.

Cardinality of Sets

How big is a set?

(2.2.8) Definition.

- Let S and T be sets. S and T have the **same cardinality**, $\mathrm{Card}(S) = \mathrm{Card}(T)$, if there is a bijective map from S to T.

Notation: Let S be a set. Then

$$\mathrm{Card}(S) = \begin{cases} 0 & \text{if } S = \emptyset; \\ n & \text{if } \mathrm{Card}(S) = \mathrm{Card}\big(\{1, 2, \ldots, b\}\big); \\ \infty & \text{otherwise.} \end{cases}$$

Note: Even if $\mathrm{Card}(S) = \infty$ and $\mathrm{Card}(T) = \infty$, one may have that $\mathrm{Card}(S) \neq \mathrm{Card}(T)$.

(2.2.9) Definition.

- A set S is **finite** if $\mathrm{Card}(S) \neq \infty$.
- A set S is **infinite** if S is not finite.
- A set S is **countable** if either S is finite or if $\mathrm{Card}(S) = \mathrm{Card}(\mathbb{P})$.
- A set S is **countably infinite** if S is countable and not finite.
- A set S is **uncountable** if S is not countable.

HW: Show that $\mathrm{Card}(\mathbb{R}) = \infty$ and $\mathrm{Card}(\mathbb{Q}) = \infty$ and that $\mathrm{Card}(\mathbb{R}) \neq \mathrm{Card}(\mathbb{Q})$.