MATH 221: Calculus and Analytic Geometry
Prof. Ram, Fall 2006

HOMEWORK 2
DUE September 18, 2006

Problem A. Angles

(1) What is π and where did it come from?

(2) Explain how to measure angles in radians, in degrees, and how convert from degrees to radians.

(3) What is the connection between measuring angles in radians and measuring distances?

(4) What is the circumference of a circle of radius r? How do you know?

(5) What is the length of an arc of angle θ on the boundary of a circle of radius r? How do you know?

(6) What is the area of a circle of radius r? How do you know?

(7) What is the area of a sector of angle θ in a circle of radius r? How do you know?

(8) Using angles, what is $\sin x$?

(9) Using angles, what is $\cos x$?

(10) Using angles, show that $\sin(-x) = -\sin x$.

(11) Using angles, show that $\cos(-x) = \cos x$.

(12) Using angles, show that $\sin^2 x + \cos^2 x = 1$.

(13) Using angles, show that $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

(14) Using angles, show that $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

Problem B. Computing trigonometric functions

(1) Explain how to derive $\sin\frac{\pi}{6}$, $\cos\frac{\pi}{6}$, $\tan\frac{\pi}{6}$, $\cot\frac{\pi}{6}$, $\sec\frac{\pi}{6}$ and $\csc\frac{\pi}{6}$ in radical form.

(2) Explain how to derive $\sin\frac{\pi}{3}$, $\cos\frac{\pi}{3}$, $\tan\frac{\pi}{3}$, $\cot\frac{\pi}{3}$, $\sec\frac{\pi}{3}$ and $\csc\frac{\pi}{3}$ in radical form.

(3) Explain how to derive $\sin\frac{\pi}{4}$, $\cos\frac{\pi}{4}$, $\tan\frac{\pi}{4}$, $\cot\frac{\pi}{4}$, $\sec\frac{\pi}{4}$ and $\csc\frac{\pi}{4}$ in radical form.

(4) Explain how to derive $\sin\frac{\pi}{2}$, $\cos\frac{\pi}{2}$, $\tan\frac{\pi}{2}$, $\cot\frac{\pi}{2}$, $\sec\frac{\pi}{2}$ and $\csc\frac{\pi}{2}$ in radical form.

(5) Explain how to derive $\sin 0$, $\cos 0$, $\tan 0$, $\cot 0$, $\sec 0$ and $\csc 0$ in radical form.

(6) Explain how to derive $\sin\frac{3\pi}{4}$, $\cos\frac{3\pi}{4}$, $\tan\frac{3\pi}{4}$, $\cot\frac{3\pi}{4}$, $\sec\frac{3\pi}{4}$ and $\csc\frac{3\pi}{4}$ in radical form.

(7) Explain how to derive $\sin\frac{-2\pi}{3}$, $\cos\frac{-2\pi}{3}$, $\tan\frac{-2\pi}{3}$, $\cot\frac{-2\pi}{3}$, $\sec\frac{-2\pi}{3}$ and $\csc\frac{-2\pi}{3}$ in radical form.

(8) Compute $\sin\frac{\pi}{6} + \cos\frac{\pi}{6}$ in radical form.

(9) Compute $\left(\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{6}\right)$ in radical form.

(10) Compute $\left(\tan\frac{\pi}{3}\right)\left(\cot\frac{\pi}{3}\right)$ in radical form.

Problem C. Trigonometric function identities

(1) Verify the identity $\dfrac{\sec A - 1}{\sec A + 1} + \dfrac{\cos A - 1}{\cos A + 1} = 0.$

(2) Verify the identity $\sin V(1 + \cot^2 V) = \csc V.$

(3) Verify the identity $\dfrac{\sin(\pi/2 - w)}{\cos(\pi/2 - w)} = \cot w.$

(4) Verify the identity $\sec(\pi/2 - z) = \dfrac{1}{\sin z}.$

(5) Verify the identity $1 + \tan^2(\pi/2 - x) = \dfrac{1}{\cos^2(\pi/2 - x)}.$

(6) Verify the identity $\dfrac{\sin A}{\csc A} + \dfrac{\cos A}{\sec A} = 1.$

(7) Verify the identity $\dfrac{\sec B}{\cos B} - \dfrac{\tan B}{\cot B} = 1.$

(8) Verify the identity $\dfrac{1}{\csc^2 w} + \sec^2 w + \dfrac{1}{\sec^2 w} = 2 + \dfrac{\sec^2 w}{\csc^2 w}.$

(9) Verify the identity $\sec^4 V - \sec^2 V = \dfrac{1}{\cot^4 V} + \dfrac{1}{\cot^2 V}.$

(10) Verify the identity $\sin^4 x + \cos^2 x = \cos^4 x + \sin^2 x.$

(11) Verify the identity $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$.

(12) Verify the identity $\cot(\alpha/2) = \dfrac{\sin\alpha}{1 - \cos\alpha}$.

(13) Verify the identity $\cos(\pi/6 - x) + \cos(\pi/6 + x) = \sqrt{3}\cos x$.

(14) Verify the identity $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$.

(15) Verify the identity $\sin(\pi/3 - x) + \sin(\pi/3 + x) = \sqrt{3}\cos x$.

(16) Verify the identity $\cos(\pi/4 - x) - \cos(\pi/4 + x) = \sqrt{2}\sin x$.

(17) Verify the identity $2\sin\alpha\cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$.

(18) Verify the identity $2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$.

Problem D. Fun trigonometric function identities

(1) Verify the identity $\cos 2\theta = 2\sin(\pi/4 + \theta)\sin(\pi/4 - \theta)$.

(2) Verify the identity $(1/2)\sin 2A = \dfrac{\tan A}{1 + \tan^2 A}$.

(3) Verify the identity $\cot(x/2) = \dfrac{1 + \cos x}{\sin x}$.

(4) Verify the identity $\sin 2B(\cot B + \tan B) = 2$.

(5) Verify the identity $\dfrac{1 - \tan^2\theta}{1 + \tan^2\theta} = \cos 2\theta$.

(6) Verify the identity $1 + \cos 2A = \dfrac{2}{1 + \tan^2 A}$.

(7) Verify the identity $\tan 2x\tan x + 2 = \dfrac{\tan 2x}{\tan x}$.

(8) Verify the identity $\csc A\sec A = 2\csc 2A$.

(9) Verify the identity $\cot x = \dfrac{\sin 2x}{1 - \cos 2x}$.

(10) Verify the identity $1 - \sin A = \left(\sin\dfrac{A}{2} - \cos\dfrac{A}{2}\right)^2$.

(11) Verify the identity $\cos^4 A = \dfrac{2\cos 2A + \cos^2 2A + 1}{4}$.

(12) Verify the identity $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}$.

(13) Verify the identity $\dfrac{\sin\alpha + \sin 3\alpha}{\cos\alpha + \cos 3\alpha} = \tan 2\alpha$.

(14) Verify the identity $\dfrac{\cos 2A}{1 + \sin 2A} = \dfrac{\cot A - 1}{\cot A + 1}$.

(15) Verify the identity $\dfrac{\cos A + \sin A}{\cos A - \sin A} = \dfrac{1 + \sin 2A}{\cos 2A}$.

(16) Verify the identity $\cot\alpha - \cot\beta = \dfrac{\sin(\beta - \alpha)}{\sin\alpha \sin\beta}$.

(17) Verify the identity $\tan\theta \csc\theta \cos\theta = 1$.

(18) Verify the identity $\cos^2\theta = \dfrac{\cot^2\theta}{1 + \cot^2\theta}$.

(19) Verify the identity $\dfrac{1 - \sin A}{1 + \sin A} = (\sec A - \tan A)^2$.

(20) Verify the identity $(\tan A - \cot A)^2 + 4 = \sec^2 A + \csc^2 A$.

(21) Verify the identity $\cos B \cos(A + B) + \sin B \sin(A + B) = \cos A$.

(22) Verify the identity $\dfrac{\tan A - \sin A}{\sec A} = \dfrac{\sin^3 A}{1 + \cos A}$.

(23) Verify the identity $\dfrac{2\tan^2 A}{1 + \tan^2 A} = 1 - \cos 2A$.

(24) Verify the identity $\tan 2A = \tan A + \dfrac{\tan A}{\cos 2A}$.

(25) Verify the identity $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$.

(26) Verify the identity $\dfrac{4\sin A}{1 - \sin^2 A} = \dfrac{1 + \sin A}{1 - \sin A} - \dfrac{1 - \sin A}{1 + \sin A}$.

(27) Verify the identity $\tan A + \sin A = \dfrac{\csc A + \cot A}{\csc A \cot A}$.

Problem E. Inverse trig function identities

(1) Verify the identity $\cos(\tan^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$.

(2) Verify the identity $\sin(\tan^{-1} x) = \dfrac{x}{\sqrt{1+x^2}}$.

(3) Verify the identity $\sin(\cos^{-1} x) = \sqrt{1-x^2}$.

(4) Verify the identity $\tan(\cos^{-1} x) = \dfrac{\sqrt{1-x^2}}{x}$.

(5) Verify the identity $\cos(\sin^{-1} x) = \sqrt{1-x^2}$.

(6) Verify the identity $\tan(\cot^{-1} x) = 1/x$.

(7) Verify the identity $\cot(\cot^{-1} 2) = 2$.

(8) Verify the identity $\sin(\cot^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$.

(9) Verify the identity $\cos(\cot^{-1} x) = \dfrac{x}{\sqrt{1+x^2}}$.

(10) Verify the identity $\sin^{-1}(-x) = -\sin^{-1} x$.

(11) Verify the identity $\tan^{-1}(-x) = -\tan^{-1} x$.

(12) Verify the identity $\tan^{-1} x = \cot^{-1}(1/x)$.

(13) Verify the identity $\tan^{-1} x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right)$.

(14) Verify the identity $\sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$.

Problem F. Basic derivatives

(1) What is $\dfrac{d}{dx}$?

(2) Explain why $\dfrac{d1}{dx} = 0$.

(3) Explain why $\dfrac{da}{dx} = 0$ if a is a number.

(4) Explain why $\dfrac{dx}{dx} = 1$.

(5) Explain why $\dfrac{dx^2}{dx} = 2x$.

(6) Explain why $\dfrac{dx^3}{dx} = 3x^2$.

(7) Explain why $\dfrac{dx^{-1}}{dx} = -x^{-2}$.

(8) Explain why $\dfrac{dx^{-2}}{dx} = -2x^{-3}$.

(9) Explain why $\dfrac{dx^{-3}}{dx} = -3x^{-4}$.

(10) Explain why $\dfrac{d(3x^2+2x)^{-1}}{dx} = \dfrac{-(6x+2)}{(3x^2+2x)^2}$.

(11) Explain why $\dfrac{dx^{1/2}}{dx} = \dfrac{1}{2}x^{-1/2}$.

(12) Explain why $\dfrac{dx^{1/3}}{dx} = \dfrac{1}{3}x^{-2/3}$.

(13) Explain why $\dfrac{dx^{3/5}}{dx} = \dfrac{3}{5}x^{-2/5}$.

(14) Explain why $\dfrac{dx^n}{dx} = nx^{n-1}$, for all positive integers n.

(15) Explain why $\dfrac{dx^n}{dx} = nx^{n-1}$, for $n = 0$.

(16) Explain why $\dfrac{dx^n}{dx} = nx^{n-1}$, for all negative integers n.

(17) Explain why $\dfrac{dx^{m/n}}{dx} = (m/n)x^{(m/n)-1}$, for all integers m and n, with $n \neq 0$.