MATH 221: Calculus and Analytic Geometry
Prof. Ram, Fall 2006

HOMEWORK 3
DUE September 25, 2006

Problem A. The chain rule

(1) Let g be a function. Show that $\dfrac{dg^0}{dx} = 0\dfrac{dg}{dx}$.

(2) Let g be a function. Show that $\dfrac{dg^1}{dx} = 1g^0\dfrac{dg}{dx}$.

(3) Let g be a function. Show that $\dfrac{dg^2}{dx} = 2g^1\dfrac{dg}{dx}$.

(4) Let g be a function. Show that $\dfrac{dg^3}{dx} = 3g^2\dfrac{dg}{dx}$.

(5) Let g be a function. Show that $\dfrac{dg^4}{dx} = 4g^3\dfrac{dg}{dx}$.

(6) Let g be a function. Show that $\dfrac{dg^5}{dx} = 5g^4\dfrac{dg}{dx}$.

(7) Let g be a function. Show that $\dfrac{dg^n}{dx} = ng^{n-1}\dfrac{dg}{dx}$ for any positive integer n.

(8) Let $f(y) = 4y^3 + 7y^2 + 2y - 13$ and let g be a function.
Show that $\dfrac{d(f(g))}{dx} = (12g^2 + 14g + 2)\dfrac{dg}{dx}$.

(9) Let f be a polynomial and let g be a function. Show that $\dfrac{d(f(g))}{dx} = \dfrac{df}{dg} \cdot \dfrac{dg}{dx}$.

Problem B. Derivatives of the basic functions.

(1) Explain why $\dfrac{de^x}{dx} = e^x$.

(2) Explain why $\dfrac{d\sin x}{dx} = \cos x$.

(3) Explain why $\dfrac{d\cos x}{dx} = -\sin x$.

(4) Explain why $\dfrac{d\tan x}{dx} = \sec^2 x$.

(5) Explain why $\dfrac{d\cot x}{dx} = -\csc^2 x$.

(6) Explain why $\dfrac{d\sec x}{dx} = \tan x \sec x$.

(7) Explain why $\dfrac{d\csc x}{dx} = -\cot x \csc x$.

(8) Explain why $\dfrac{d\ln x}{dx} = \dfrac{1}{x}$.

(9) Explain why $\dfrac{d\sin^{-1} x}{dx} = \dfrac{1}{\sqrt{1-x^2}}$.

(10) Explain why $\dfrac{d\cos^{-1} x}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$.

(11) Explain why $\dfrac{d\tan^{-1} x}{dx} = \dfrac{1}{1+x^2}$.

(12) Explain why $\dfrac{d\cot^{-1} x}{dx} = -\dfrac{1}{1+x^2}$.

(13) Explain why $\dfrac{d\csc^{-1} x}{dx} = -\dfrac{1}{|x|\sqrt{x^2-1}}$.

Problem C. Computing some derivatives

(1) Find $\dfrac{dy}{dx}$ when $y = (2x+3)(5x+6)$.

(2) Find $\dfrac{dy}{dx}$ when $y = \left(x + \dfrac{1}{x}\right)\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)$.

(3) Find $\dfrac{dy}{dx}$ when $y = (2x-5)^2(3x-4)^3$.

(4) Find $\dfrac{dy}{dx}$ when $y = \left(ex^2 + \dfrac{\pi}{x^3} + x^{7/2}\right)$.

(5) Find $\dfrac{dy}{dx}$ when $y = \left(\dfrac{x-3}{x-4}\right)^2$.

(6) Find $\dfrac{dy}{dx}$ when $y = \dfrac{3x+5}{4-x^2}$.

(7) Find $\dfrac{dy}{dx}$ when $y = \dfrac{x}{\sqrt{1-2x}}$.

(8) Find $\dfrac{dy}{dx}$ when $y = \dfrac{1+\sqrt{x}}{1-\sqrt{x}}$.

(9) Find $\dfrac{dy}{dx}$ when $y = \dfrac{2(x+1)}{x^2+2x-3}$.

(10) Find $\dfrac{dy}{dx}$ when $y = \dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$.

(11) Find $\dfrac{dy}{dx}$ when $y = \dfrac{x^2-2}{x+1}$.

(12) Find $\dfrac{dy}{dx}$ when $y = \dfrac{\sqrt{x}}{\sqrt{x-3}}$.

(13) Find $\dfrac{dy}{dx}$ when $y = \dfrac{x^n+1}{x^n-1}$.

(14) Find $\dfrac{dy}{dx}$ when $y = \dfrac{\sqrt{1+x^2}}{\sqrt{1-x^2}}$.

(15) Find $\dfrac{dy}{dx}$ when $y = \dfrac{2x^2-1}{x\sqrt{1+x^2}}$

(16) Find $\dfrac{dy}{dx}$ when $y = u^n$.

(17) Find $\dfrac{dy}{dx}$ when $y = \sqrt{1-x^2}$.

Problem D. Correcting derivative identities

(1) Correct the identity $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} + u\dfrac{dv}{dx}}{v^2}$.

(2) Correct the identity $\dfrac{d}{dx}(u+v) = \dfrac{du}{dx} - \dfrac{dv}{dx}$.

(3) Correct the identity $\dfrac{d}{dx}(uv) = \dfrac{du}{dx} \cdot \dfrac{dv}{dx}$.

Problem E. Verifying derivative identities

(1) If $y = x^{7/2}$ show that $2x\dfrac{dy}{dx} - 7y = 0$.

(2) If $y = 3 - x^2$ prove that $\left(\dfrac{dy}{dx}\right)^2 - 4x^2 = 0$.

(3) If $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$ show that $2x\dfrac{dy}{dx} + y - 2\sqrt{x} = 0$.

(4) If $y = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!}$ show that $\dfrac{dy}{dx} - y + \dfrac{x^n}{n!} = 0$.

(5) If $y = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ show that $\dfrac{dy}{dx} = y$.

(6) If $z = \dfrac{3}{1+t}$ show that $3t\dfrac{dz}{dt} = z(z-3)$.

(7) If $y = \dfrac{1}{a-z}$ show that $\dfrac{dz}{dy} = (z-a)^2$.

(8) If $y = \dfrac{x}{x-p}$ prove that $x\dfrac{dy}{dx} = y(1-y)$.

(9) If $y = x - \sqrt{1+x^2}$ show that $(1+x^2)\left(\dfrac{dy}{dx}\right)^2 = y^2$.

(10) If $y = x^2$ show that $\left(\dfrac{dy}{dx}\right)^2 = 4y$.

(11) If $y = \sqrt{1+x^5}$ show that $\dfrac{dy}{dx} = \dfrac{5x^4}{2y}$.

Problem F. Derivatives at a point

(1) Find $\dfrac{dy}{dx}$ at $x = 3$ when $y = x^6 + 3x^2 + 5$.

(2) Find $\left.\dfrac{dy}{dx}\right|_{x=3}$ when $y = (x+1)(x+2)$.

Problem G. Derivatives with respect to functions

(1) Differentiate $t^2 - \frac{4}{t^2}$ with respect to t^5.

(2) Differentiate $\frac{x^2}{1+x^2}$ with respect to x^2.

(3) Differentiate $\frac{ax+b}{cx+d}$ with respect to $\frac{a_1x+b_1}{c_1x+d_1}$.

(4) Differentiate x^3 with respect to x^2.

(5) Differentiate $\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$ with respect to $\sqrt{1-x^4}$.

(6) Differentiate $\frac{x}{1+x^2}$ with respect to x^3.

(7) Differentiate $x - \sqrt{1-x^2}$ with respect to $\sqrt{1-x^2}$.

(8) Differentiate $7x^5 - 11x^2$ with respect to $7x^2 - 15x$.

Problem H. Derivatives of parametric equations

(1) Find $\frac{dy}{dx}$ when $x = pt$ and $y = p/t$.

(2) Find $\frac{dy}{dx}$ when $x = at^2$ and $y = 2at$.

(3) Find $\frac{dy}{dx}$ when $y = \frac{2at^2}{1+t^2}$ and $x = \frac{2a}{1+t^2}$.

(4) Find $\frac{dy}{dx}$ when $x = a\frac{1-t^2}{1+t^2}$ and $y = b\frac{2t}{1+t^2}$.

(5) Find $\frac{dy}{dx}$ when $x = a\sqrt{\frac{t^2-1}{t^2+1}}$ and $y = at\sqrt{\frac{t^2-1}{t^2+1}}$.

(6) Find $\frac{dy}{dx}$ when $x = a\frac{1+t^2}{1-t^2}$ and $y = \frac{2bt}{1-t^2}$.

(7) Find $\frac{dy}{dx}$ when $x = \frac{3at}{1+t^3}$ and $y = \frac{3at^2}{1+t^3}$.

(8) Find $\dfrac{dy}{dx}$ when $x = \dfrac{1-t^2}{1+t^2}$ and $y = \dfrac{2t}{1+t^2}$.

Problem I. Implicit differentiation

(1) Find $\dfrac{dy}{dx}$ when $x^4 + y^4 = 4a^2x^2y^2$.

(2) Find $\dfrac{dy}{dx}$ when $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(3) Find $\dfrac{dy}{dx}$ when $x^5 + y^5 - 5ax^2y^2 = 0$.

(4) If $ax^2 + by^2 + 2gx + 2fy + 2hxy + c = 0$ show that $\dfrac{dy}{dx} + \dfrac{ax + hy + g}{hx + by + f} = 0$.

(5) If $xy + px + q = 0$ prove that $x^2\dfrac{dy}{dx}$ is always constant.

(6) Find $\dfrac{dy}{dx}$ when $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Problem J. Derivatives with trigonometric functions.

(1) Find $\dfrac{dy}{dx}$ when $y = \sin(3x + 2)$.

(2) Find $\dfrac{dy}{dx}$ when $y = \sqrt{\sin x^4}$.

(3) Find $\dfrac{dy}{dx}$ when $y = x^2 \sin x$.

(4) Find $\dfrac{dy}{dx}$ when $y = \tan x \sin 2x$.

(5) Find $\dfrac{dy}{dx}$ when $y = \sin x^2 - \dfrac{\tan x}{1 + x^2}$.

(6) Find $\dfrac{dy}{dx}$ when $y = \dfrac{2\cos x - x}{x + 2}$.

(7) Find $\dfrac{dy}{dx}$ when $y = (1 + x^2) + \dfrac{x}{\sin x}$.

(8) Find $\dfrac{dy}{dx}$ when $y = \dfrac{\sin 2x}{\cos x}$.

(9) Find $\frac{dy}{dx}$ when $y = \sin(x/3)\csc(2x/3)$.

(10) Find $\frac{dy}{dx}$ when $y = \sin(\sin x + \cos x)$.

(11) Find $\frac{dy}{dx}$ when $y = \sqrt{\sec^2 x + \csc^2 x}$.

(12) Find $\frac{dy}{dx}$ when $y = (x^2 - 1)\left(\cot x + \frac{\tan x}{1 + x^2}\right)$.

(13) Find $\frac{dy}{dx}$ when $y = \sqrt{\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}}$.

(14) Find $\frac{dy}{dx}$ when $y = \frac{\sec x + \tan x}{\sec x - \tan x}$.

(15) Find $\frac{dy}{dx}$ when $y = \sqrt{\frac{1 - \cos x}{1 + \cos x}}$.

(16) Find $\frac{dy}{dx}$ when $y = x^3 \tan^2(x/2)$.

(17) If $y = \tan(\cos(\sin\theta))$ find dy/dx.

Problem K. Derivatives with exponentials and logs.

(1) Find $\frac{dy}{dx}$ when $y = \left(ex^2 + \frac{\pi}{x^3} + x^{7/2}\right)$.

(2) Find $\frac{dy}{dx}$ when $y = a^{ax+b}$.

(3) Find $\frac{dy}{dx}$ when $y = a^{x^3}$.

(4) Find $\frac{dy}{dx}$ when $y = 6^{2x}$.

(5) Find $\frac{dy}{dx}$ when $y = \ln(ax^2 + b)$.

(6) Find $\frac{dy}{dx}$ when $y = e^{3\ln x}$.

(7) Find $\frac{dy}{dx}$ when $y = e^{2x} - e^{-2x}$.

(8) Find $\frac{dy}{dx}$ when $y = e^{x^2+2x}$.

(9) Find $\frac{dy}{dx}$ when $y = a^x x^a$.

(10) Find $\frac{dy}{dx}$ when $y = xe^x$.