MATH 221: Calculus and Analytic Geometry
Prof. Ram, Fall 2006

HOMEWORK 4
DUE October 2, 2006

Problem A. Expansions

For questions 1-9 suppose that

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots.$$

(1) Show that $c_0 = f(a)$.

(2) Show that $c_1 = \frac{df}{dx}\Big|_{x=a}$.

(3) Show that $c_2 = \frac{1}{2}\left(\frac{d^2f}{dx^2}\Big|_{x=a}\right)$.

(4) Show that $c_3 = \frac{1}{3!}\left(\frac{d^3f}{dx^3}\Big|_{x=a}\right)$.

(5) Show that $c_4 = \frac{1}{4!}\left(\frac{d^4f}{dx^4}\Big|_{x=a}\right)$.

(6) Show that $c_5 = \frac{1}{5!}\left(\frac{d^5f}{dx^5}\Big|_{x=a}\right)$.

(7) Explain why $c_n = \frac{1}{n!}\left(\frac{d^nf}{dx^n}\Big|_{x=a}\right)$.

(8) Show that

$$f(a+\Delta x) = f(a) + \left(\frac{df}{dx}\Big|_{x=a}\right)\Delta x + \frac{1}{2}\left(\frac{d^2f}{dx^2}\Big|_{x=a}\right)(\Delta x)^2$$

$$+ \frac{1}{3!}\left(\frac{d^3f}{dx^3}\Big|_{x=a}\right)(\Delta x)^3 + \frac{1}{4!}\left(\frac{d^4f}{dx^4}\Big|_{x=a}\right)(\Delta x)^4 + \cdots.$$

(9) Show that $\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = \frac{df}{dx}\Big|_{x=a}$.

(10) Give a series expansion for e^x.

(11) Give a series expansion for $\sin x$.

(12) Give a series expansion for $\cos x$.

(13) Give a series expansion for $\dfrac{1}{1-x}$.

(14) Give a series expansion for $\dfrac{1}{1+x}$.

(15) Give a series expansion for $\dfrac{1}{1+x^2}$.

(16) Explain why $1+\dfrac{1}{3}+\dfrac{1}{3^2}+\dfrac{1}{3^3}+\dfrac{1}{3^4}+\cdots=\dfrac{3}{2}$.

(17) Explain why $1+\dfrac{1}{3}+\dfrac{1}{3^2}+\dfrac{1}{3^3}+\dfrac{1}{3^4}+\cdots+\dfrac{1}{3^{50}}=\dfrac{3}{2}-\dfrac{1}{2\cdot 3^{50}}$.

Problem B. Derivatives at a point.

(1) Let $y=\tan 2x-2\tan x+2$. Find $\dfrac{dy}{dx}$ at $x=\pi/4$.

(2) Let $y=\dfrac{\sin^2 x+\cos x}{1+x^2}$. Find $\left.\dfrac{dy}{dx}\right|_{x=0}$ and $\left.\dfrac{dy}{dx}\right|_{x=\pi/2}$.

(3) Let $y=\cos(\sin x^2)$. Find $\left.\dfrac{dy}{dx}\right|_{x=\pi/3}$.

(4) Let $y=(\cot\sqrt{x}+5\sin^2\sqrt{x})^2$. Find $\dfrac{dy}{dx}$ at $x=\pi^2/16$.

(5) Let $y=\dfrac{\sin x^2}{\sqrt{1+x^2}}$. Find $\left.\dfrac{dy}{dx}\right|_{x=0}$ and $\left.\dfrac{dy}{dx}\right|_{x=\sqrt{\pi/2}}$.

Problem C. Differential equations.

(1) If $y=x+\tan x$ show that $\cos^2 x\cdot\dfrac{d^2y}{dx^2}-2y+2x=0$.

(2) If $y=A\cos nx+B\sin nx$ show that $\dfrac{d^2y}{dx^2}+n^2y=0$.

(3) If $y = 2\sin x + 3\cos x$ show that $y + \dfrac{d^2y}{dx^2} = 0$.

(4) If $y = a\sin x + b\cos x$ show that $\dfrac{d^2y}{dx^2} + y = 0$.

(5) If $y = \sin(\sin x)$ show that $\dfrac{d^2y}{dx^2} + \tan x\dfrac{dy}{dx} + y\cos^2 x = 0$.

(6) If $y = a\sin x + b\cos x$ prove that $y^2 + \left(\dfrac{dy}{dx}\right)^2 = a^2 + b^2$.

(7) If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \sqrt{\cdots}}}}$ show that $(2y-1)\dfrac{dy}{dx} = \cos x$.

Problem D. Parametric equations.

(1) Find $\dfrac{dy}{dx}$ when $x = a\cos\theta$ and $y = b\sin\theta$.

(2) Find $\dfrac{dy}{dx}$ when $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$.

(3) Find $\dfrac{dy}{dx}$ when $x = a\sec^2\theta$ and $y = b\tan^3\theta$.

(4) Find $\dfrac{dy}{dx}$ when $x = b\sin^3\phi$ and $y = b\cos^3\phi$.

(5) If $x = a(t - \sin t)$ and $y = a(1 - \cos t)$ find $\dfrac{d^2y}{dx^2}$.

(6) If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$ find $\dfrac{d^2y}{dx^2}$ at $\theta = \pi/2$.

(7) If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$ find $\dfrac{d^2y}{dx^2}$ at $\theta = \pi/2$.

(8) If $x = \sin t$ and $y = \sin mt$ prove that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + m^2y = 0$.

Problem E. Implicit differentiation.

(1) Find $\dfrac{dy}{dx}$ when $y^2\sin x + y\tan x + (1+x^2)\cos x = 0$.

(2) Find $\dfrac{dy}{dx}$ when $\sin(xy) + \dfrac{x}{y} = x^2 - y$.

(3) Find $\dfrac{dy}{dx}$ when $2y^2 + \dfrac{y}{1+x^2} + \tan^2 x + \sin y = 0$.

(4) Find $\dfrac{dy}{dx}$ when $\tan(x+y) + \tan(x-y) = 1$.

(5) Find $\dfrac{dy}{dx}$ when $a\sin(xy) + b\cos(x/y) = 0$.

(6) If $x = \ln(\tan(y/x))$ find $\dfrac{dy}{dx}$.

Problem F. Derivatives with inverse trig functions.

(1) Find $\dfrac{dy}{dx}$ when $y = \sin^{-1} x^3$.

(2) Find $\dfrac{dy}{dx}$ when $y = \tan^{-1} \sqrt{x}$.

(3) Find $\dfrac{dy}{dx}$ when $y = \sin^{-1} 3x$.

(4) Find $\dfrac{dy}{dx}$ when $y = \csc^{-1} x^2$.

(5) Find $\dfrac{dy}{dx}$ when $y = \cos^{-1} \sqrt{x}$.

(6) Find $\dfrac{dy}{dx}$ when $y = \csc^{-1}(\sin x)$.

(7) Find $\dfrac{dy}{dx}$ when $y = \tan^{-1} \sqrt{x-1}$.

(8) Find $\dfrac{dy}{dx}$ when $y = \sin(\tan^{-1} x)$.

(9) Find $\dfrac{dy}{dx}$ when $y = x\cos^{-1} x$.

(10) Find $\dfrac{dy}{dx}$ when $y = x\sin^{-1} x$.

(11) Find $\dfrac{dy}{dx}$ when $y = \tan^{-1} \sqrt{x} - \tan^{-1} x$.

(12) Find $\dfrac{dy}{dx}$ when $y = (1 + x^2)\tan^{-1} x$.

(13) Find $\dfrac{dy}{dx}$ when $y = \tan x \cos^{-1} x$.

(14) Find $\dfrac{dy}{dx}$ when $y = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) + \tan^{-1} x$.

(15) Find $\dfrac{dy}{dx}$ when $y = (1 - x^2)\cos^{-1} x$.

(16) Find $\dfrac{dy}{dx}$ when $y = \tan x \cdot \tan^{-1} x$.

(17) Find $\dfrac{dy}{dx}$ when $y = \sec^{-1} x + \csc^{-1} x$.

(18) Find $\dfrac{dy}{dx}$ when $y = \tan^{-1}(a/x) \cdot \cot^{-1}(x/a)$.

(19) Find $\dfrac{dy}{dx}$ when $y = (\tan^{-1} 2x)^3$.

(20) Find $\dfrac{dy}{dx}$ when $y = \cos^{-1}(\tan x^2)$.

(21) Find $\dfrac{dy}{dx}$ when $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$.

(22) Find $\dfrac{dy}{dx}$ when $y = \sec^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$.

(23) Find $\dfrac{dy}{dx}$ when $y = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$.

(24) Find $\dfrac{dy}{dx}$ when $y = \tan^{-1}\left(\dfrac{1+x^2}{1-x^2}\right)$.

(25) Find $\dfrac{dy}{dx}$ when $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$.

(26) Find $\dfrac{dy}{dx}$ when $y = \cot^{-1}\left(\dfrac{1+\cos x}{1-\cos x}\right)^{1/2}$.

(27) Find $\dfrac{dy}{dx}$ when $y = \cot^{-1}\left(\dfrac{1+\cos 3x}{1-\cos 3x}\right)^{1/2}$.

(28) Find $\dfrac{dy}{dx}$ when $y = \sin^{-1}\left(\dfrac{a + b\cos x}{b + a\cos x}\right)$.

(29) Find $\dfrac{dy}{dx}$ when $y = \cos^{-1}\left(\dfrac{1 + 2\cos x}{2 + \cos x}\right)$.

(30) Find $\dfrac{dy}{dx}$ when $y = \tan^{-1}\left(\dfrac{1 - \cos x}{\sin x}\right)$.

(31) Differentiate $\sin^{-1}\left(\dfrac{x^2 - 1}{1 + x^2}\right)$ with respect to $\cos^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right)$.

(32) If $y = \dfrac{\sin^{-1} x}{\sqrt{1 - x^2}}$ prove that $(1 - x^2)\dfrac{dy}{dx} - xy = 1$.

Problem G. Derivatives with trigonometric functions.

(1) Find $\dfrac{dy}{dx}$ when $y = x\cos x - \sin x$.

(2) Find $\dfrac{dy}{dx}$ when $y = \cos^3 3x$.

(3) Find $\dfrac{dy}{dx}$ when $y = (x^2 + \cos x)^4$.

(4) Find $\dfrac{dy}{dx}$ when $y = \sin x \sin 2x$.

(5) Find $\dfrac{dy}{dx}$ when $y = \dfrac{\sin 2x}{x^2}$.

Problem H. Derivatives with exponentials and logs.

(1) Find $\dfrac{dy}{dx}$ when $y = \ln\left(x + \sqrt{x^2 + a^2}\right)$.

(2) Find $\dfrac{dy}{dx}$ when $y = \dfrac{1 + e^x}{1 - e^x}$.

(3) Find $\dfrac{dy}{dx}$ when $y = \ln\left(\dfrac{x^2 + x + 1}{x^2 - x - 1}\right)$.

(4) Find $\dfrac{dy}{dx}$ if $y = \ln\left[e^x\left(\dfrac{x - 2}{x + 2}\right)^{3/4}\right]$.

(5) Find $\dfrac{dy}{dx}$ when $y = \ln\ln\ln x^4$.

Problem I. Derivatives with exponentials, logs and trig functions.

(1) Find $\dfrac{dy}{dx}$ when $y = a^{\cos 1}$.

(2) Find $\dfrac{dy}{dx}$ when $y = \ln \dfrac{\sin^m x}{\cos^n x}$.

(3) Find $\dfrac{dy}{dx}$ when $\quad y = e^{ax}\sin bx$.

(4) Find $\dfrac{dy}{dx}$ when $y = \ln\left(\dfrac{1-\cos x}{1+\cos x}\right)$.

(5) Find $\dfrac{dy}{dx}$ when $y = \ln\sqrt{\dfrac{1-\tan x}{1+\tan x}}$.

(6) Find $\dfrac{dy}{dx}$ when $y = e^{ax}\cos(bx+c)$.

(7) Find $\dfrac{dy}{dx}$ when $y = \dfrac{\sqrt{x+\ln\tan x}}{xe^{2x}}$.

(8) Find $\dfrac{dy}{dx}$ when $y = \ln\dfrac{1+x\sin x}{1-x\sin x}$.

(9) Find $\dfrac{dy}{dx}$ when $y = \ln\left(\dfrac{1-\cos x}{1+\cos x}\right)^{1/2}$.

(10) Find $\dfrac{dy}{dx}$ when $y = \ln\sqrt{\dfrac{1+\sin x}{1-\sin x}}$.

(11) If $y = \ln(\sin x)$ show that $\dfrac{d^3y}{dx^3} = 2\csc^2 x\cot x$.

(12) If $y = e^{ax}\cos bx$ show that $\dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx} + (a^2+b^2)y = 0$.

(13) If $y = a\cos(\ln x) + b\sin(\ln x)$ show that $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$.

(14) If $y = Ae^{-kt}\cos(pt+c)$ show that $\dfrac{d^2y}{dt^2} + 2k\dfrac{dy}{dt} + n^2y = 0$, where $n^2 = p^2+k^2$.

(15) If $y = e^{-x}\cos x$ prove that $\dfrac{d^4y}{dx^4} + 4y = 0$.