MATH 221 Lecture 11, October 2, 2000 ①

$\lim_{x \to 2} f(x) = 10$ if $f(x)$ gets closer and closer to 10 as x gets closer and closer to 2.

Example Evaluate $\lim_{x \to 2} \frac{3x^2+8}{x^2-x}$.

When $x = 1$, $\frac{3x^2+8}{x^2-x} = 11$.

When $x = 1.5$, $\frac{3x^2+8}{x^2-x} = 19.66\ldots$

When $x = 1.9$, $\frac{3x^2+8}{x^2-x} = 11.011\ldots$

When $x = 1.99$, $\frac{3x^2+8}{x^2-x} = 10.091\ldots$

When $x = 1.999$, $\frac{3x^2+8}{x^2-x} = 10.00901\ldots$

When $x = 1.9999$, $\frac{3x^2+8}{x^2-x} = 10.0009001\ldots$

So $\lim_{x \to 2} \frac{3x^2+8}{x^2-x} = 10$

Usually determining the limit is straightforward

Example $\lim_{x \to 1} 6x^2 - 4x + 3 = 5$

But sometimes... ②

Example $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} \stackrel{?}{=} \frac{0}{0}$

$\frac{0}{0}$ makes NO SENSE.

Example $\lim_{x \to 0} \frac{5x}{x} \stackrel{?}{=} \frac{0}{0}$

$\lim_{x \to 0} \frac{5x}{x} = \lim_{x \to 0} 5 = 5$.

Example $\lim_{x \to 0} \frac{17x}{2x} \stackrel{?}{=} \frac{0}{0}$.

$\lim_{x \to 0} \frac{17x}{2x} = \lim_{x \to 0} \frac{17}{2} = \frac{17}{2}$.

Let's go back to

Example Evaluate $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \to 0} \frac{(\sqrt{1+x} - 1)}{x} \frac{(\sqrt{1+x} + 1)}{(\sqrt{1+x} + 1)}$$

$$= \lim_{x \to 0} \frac{1+x-1}{x(\sqrt{1+x}+1)} = \lim_{x \to 0} \frac{x}{x(\sqrt{1+x}+1)}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{1+x}+1} = \lim_{x \to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}.$$

So, whenever a limit looks like it is coming out to $\frac{0}{0}$ it needs to be looked at in a different way to see what it is really getting closer and closer to. ③

Example $\lim_{x\to 7} \frac{x^2-49}{x-7} = \lim_{x\to 7} \frac{(x-7)(x+7)}{(x-7)}$

$= \lim_{x\to 7} (x+7) = 7+7 = 14.$

Example Evaluate $\lim_{x\to 5} \frac{x^5-3125}{x-5}.$

$$\lim_{x\to 5} \frac{x^5-3125}{x-5} = \lim_{x\to 5} \frac{x^5-5^5}{x-5}$$

$$= \lim_{x\to 5} \frac{(x-5)(x^4+5x^3+5^2x^2+5^3x+5^4)}{x-5}$$

$$= \lim_{x\to 5} x^4+5x^3+5^2x^2+5^3x+5^4$$

$$= 5^4+5^4+5^4+5^4+5^4 = 5^5 = 3125.$$

Example Evaluate $\lim_{x\to a} \frac{x^{5/2}-a^{5/2}}{x-a}.$

④

$$\lim_{x\to a} \frac{x^{5/2}-a^{5/2}}{x-a} = \lim_{x\to a} \frac{(x^{5/2}-a^{5/2})}{(x-a)} \frac{(x^{5/2}+a^{5/2})}{(x^{5/2}+a^{5/2})}$$

$$= \lim_{x\to a} \frac{x^5-a^5}{x-a} \cdot \frac{1}{x^{5/2}+a^{5/2}}$$

$$= \lim_{x\to a} \frac{(x-a)(x^4+ax^3+a^2x^2+a^3x+a^4)}{x-a} \quad \frac{1}{x^{5/2}+a^{5/2}}$$

$$= \lim_{x\to a} \frac{x^4+ax^3+a^2x^2+a^3x+a^4}{x^{5/2}+a^{5/2}}$$

$$= \frac{a^4+a^4+a^4+a^4+a^4}{a^{5/2}+a^{5/2}} = \frac{5a^4}{2a^{5/2}} = \frac{5}{2} a^{3/2}.$$

Particularly useful limits

(a) $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots}{x}$

$$= \lim_{x\to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots$$

$$= 1-0+0-0+0-\cdots = 1.$$

(b) $\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)-1}{x}$

(5)

$$= \lim_{x\to 0} \frac{-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots}{x}$$

$$= \lim_{x\to 0} -\frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \frac{x^7}{8!} - \cdots$$

$$= 0 + 0 + 0 + \cdots = 0.$$

(c) $$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - 1}{x}$$

$$= \lim_{x\to 0} \frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}{x}$$

$$= \lim_{x\to 0} 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots = 1 + 0 + 0 + \cdots$$

$$= 1.$$

(d) Evaluate $\lim_{x\to 0} \frac{\ln(1+x)}{x}$.

Let $y = \ln(1+x)$. Then $e^y = 1+x$ and $x = e^y - 1$.

Also $y \to 0$ as $x \to 0$. So

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{y\to 0} \frac{y}{e^y - 1} = \lim_{y\to 0} \frac{1}{\frac{e^y - 1}{y}}$$

$$= \frac{1}{1} = 1.$$

(6)

<u>Example</u> Evaluate $\lim_{x\to 0} (1+x)^{\frac{1}{x}}$.

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = \lim_{x\to 0} \left(e^{\ln(1+x)}\right)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x)}$$

$$= \lim_{x\to 0} e^{\frac{\ln(1+x)}{x}} = e^1 = e.$$

<u>Example</u> Evaluate $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

Let $x = \frac{1}{n}$. Then $x \to 0$ as $n \to \infty$.

So

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e.$$

<u>Note:</u> $n \to \infty$ <u>means</u> n gets larger and larger.

<u>Example</u> Evaluate $\lim_{x\to\pi} \frac{\sin x}{x - \pi}$

Let $y = x - \pi$. Then $y \to 0$ as $x \to \pi$.

So

$$\lim_{x\to\pi} \frac{\sin x}{x-\pi} = \lim_{y\to 0} \frac{\sin(y+\pi)}{y} = \lim_{y\to 0} \frac{\sin y \cos\pi + \cos y \sin\pi}{y}$$

$$\cancel{= \lim_{y\to 0} \frac{0 + (-1)\cos y}{}} = \lim_{y\to 0} \frac{\sin y(-1) + \cos y \cdot 0}{y}$$

$$= \lim_{y\to 0} \frac{-\sin y}{y} = -1.$$