MATH 221 Lecture 18, October 18, 2000

Rolle's Theorem and the Mean Value Theorem

Rolle's Theorem:

If $f(a) = f(b)$, and

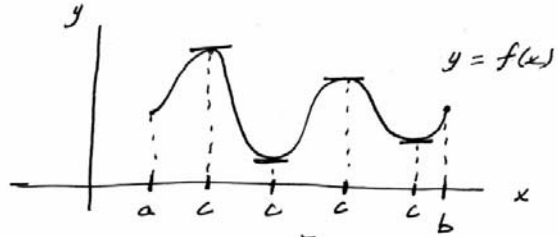
$f(x)$ is continuous between a and b, and

$f(x)$ is differentiable between a and b,

Then

there is a point c between a and b

such that $\left.\frac{df}{dx}\right|_{x=c} = 0$



4 possible choices for c!

Mean Value Theorem

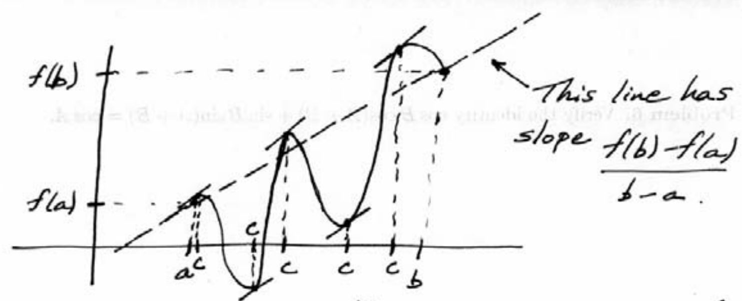
If $f(x)$ is continuous between a and b, and

$f(x)$ is differentiable between a and b,

Then there is a point c between a and b

such that

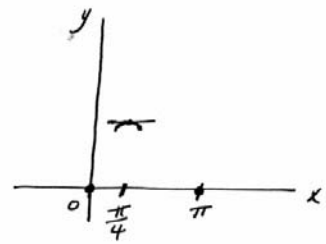
$$\left.\frac{df}{dx}\right|_{x=c} = \frac{f(b) - f(a)}{b - a}$$



5 possible choices for c.

Note: If $f(a) = f(b)$ then the line connecting $(a, f(a))$ and $(b, f(b))$ has slope 0 and so Rolle's theorem is a special case of the mean value theorem.

Example Verify Rolle's theorem for the function $f(x) = e^{-x}\sin x$ on the interval $[0, \pi]$.



$$f(0) = e^{0}\sin 0 = 0$$

$$f(\pi) = e^{\pi}\sin\pi = 0$$

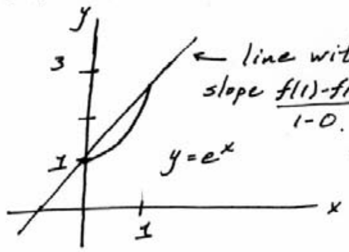
$$\frac{df}{dx} = e^{-x}\cos x - e^{-x}\sin x$$

$$= e^{-x}(\cos x - \sin x).$$

So, if $\frac{df}{dx} = 0$ then $\cos x - \sin x = 0$. So $\cos x = \sin x$

So $x = \frac{\pi}{4}$. So $c = \frac{\pi}{4}$ when $\left.\frac{df}{dx}\right|_{x=c} = 0$.

<u>Example</u> Verify the mean value theorem for $f(x) = e^x$ in the interval $[0,1]$.



$f(0) = e^0 = 1$

$f(1) = e^1 = e$

So $\dfrac{f(1)-f(0)}{1-0} = \dfrac{e-1}{1-0} = e-1$.

$\dfrac{df}{dx} = e^x$ and we want c so that $\left.\dfrac{df}{dx}\right|_{x=c} = e-1$.

If $\left.\dfrac{df}{dx}\right|_{x=c} = e^c = e-1$. Then

$c = \ln(e-1) \approx \ln(1.78)$ which is between 0 and 1.

<u>Example</u> Consider the mean value theorem for $f(x) = \frac{1}{x}$ in the interval $[-1,1]$.

$f(1) = \frac{1}{1} = 1$ and $f(-1) = \frac{1}{-1} = -1$

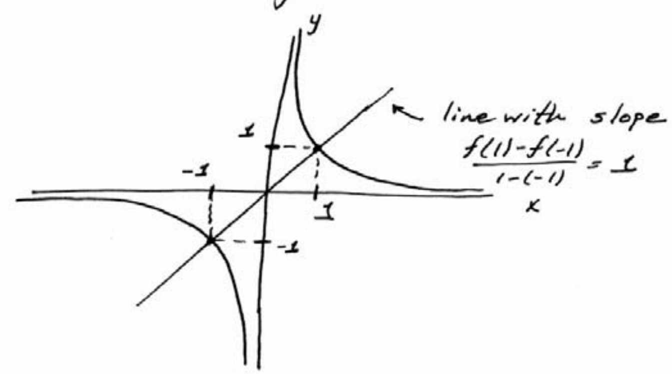
So $\dfrac{f(1)-f(-1)}{1-(-1)} = \dfrac{1-(-1)}{2} = \dfrac{2}{2} = 1$.

So we want c so that $\left.\dfrac{df}{dx}\right|_{x=c} = \dfrac{f(1)-f(-1)}{1-(-1)} = 1$

$\left.\dfrac{df}{dx}\right|_{x=c} = \left.\dfrac{d\,\frac{1}{x}}{dx}\right|_{x=c} = \left.\dfrac{-1}{x^2}\right|_{x=c} = \dfrac{-1}{c^2}$.

Find c so that $\frac{-1}{c^2} = 1$. (IMPOSSIBLE with real numbers)

What went wrong?



$f(x)$ is not continuous or differentiable at $x=0$!!

So the mean value theorem <u>does</u> <u>not</u> apply.

<u>Example</u> Show that the equation $x^5+10x+3=0$ has exactly one real root.

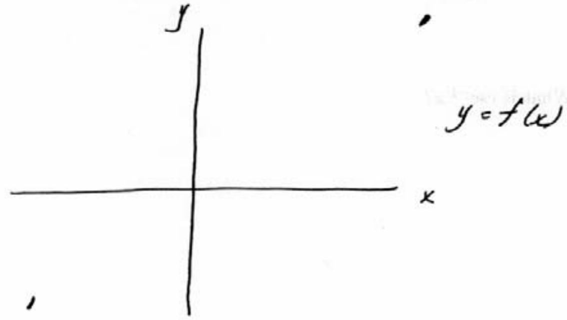
Let $f(x) = x^5+10x+3$.

We have to show that there is <u>only</u> one real number that can be plugged into $f(x)$ to get 0.

<u>Notes</u>:

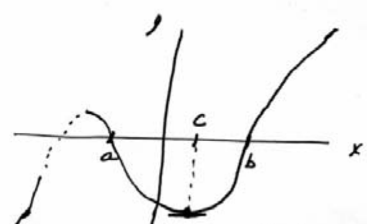
(a) As $x \to \infty$, $f(x) \to \infty$

(b) As $x \to -\infty$, $f(x) \to -\infty$.



(a) and (b) tell us that y=f(x) must cross the x-axis.

Suppose it crosses twice, at $x=a$ and $x=b$



Then $\left.\frac{df}{dx}\right|_{x=c} = 0$ for some c between a and b

So $0 = \left.\frac{df}{dx}\right|_{x=c} = \left.\frac{d(x^5+10x+3)}{dx}\right|_{x=c} = \left.5x^4+10\right|_{x=c} = 5c^4+10.$

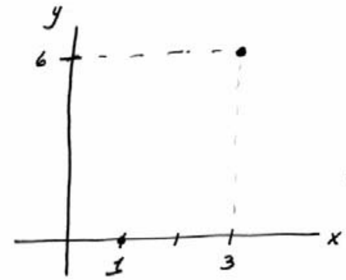
But $5c^4+10$ is never 0, no matter what c is.

So $y=f(x)$ can't cross the x-axis twice.

So it must cross it only once.

So there is exactly one number (real number) that can be plugged into $f(x)$ to get 0.

Example Discuss Rolle's theorem for $f(x) = (x-1)(2x-3)$ in the interval $1 \le x \le 3$.



$f(1) = 0$

$f(3) = (3-1)(6-3) = 6.$

Since $f(1) \neq f(3)$ we can't apply Rolle's theorem with $x=1$ and $x=3$.

Are there two points in the interval $[1,3]$ where $f(a) = f(b)$? Yes, $f(1) = 0$ and $f(\frac{3}{2}) = 0$.

So we should be able to find c between 1 and $3/2$ so that $\left.\frac{df}{dx}\right|_{x=c} = 0$.

$$\left.\frac{df}{dx}\right|_{x=c} = \left.(x-1)\cdot 2 + 1\cdot(2x-3)\right|_{x=c} = \left.4x-5\right|_{x=c} = 4c-5.$$

So $\left.\frac{df}{dx}\right|_{x=c} = 0$ when $c = \frac{5}{4}$, and $\frac{5}{4}$ is between 1 and $3/2$.