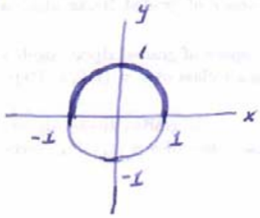
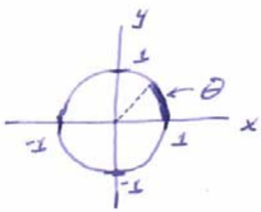
MATH 221 Lecture 2, September 8, 2000

Angles



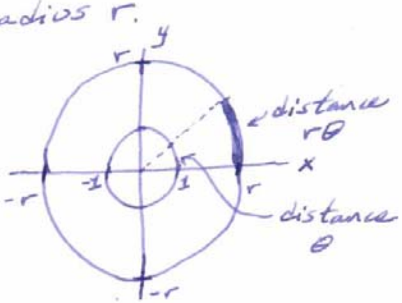
π is the distance half way around a circle of radius 1.

Measure angles according to the distance traveled on a circle of radius 1



The angle θ is measured by traveling a distance θ on a circle of radius 1.

Stretch both x and y to get a circle of radius r.



The distance θ stretches to $r\theta$

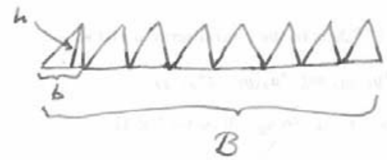
The distance 2π around a circle of radius 1 stretches to $2\pi r$ around a circle of radius r.

So the circumference of a circle is $2\pi r$, if the circle is radius r.

To find the area of a circle first approximate with a polygon inscribed in the circle.

The eight triangles form an octagon P_8 in the circle. The area of the octagon P_8 is almost the same as the area of the circle.

Unwrap the octagon



The area of the octagon is the area of the 8 triangles. The area of each triangle is $\frac{1}{2}bh$. So the area of the octagon is $\frac{1}{2}Bh$.

Take the limit as the number of triangles in the interior polygon gets larger and larger (the polygon gets closer and closer to being the circle). Then

Area of the circle $= \lim\limits_{n\to\infty}$ (area of an n-sided polygon P_n) ③

$$= \lim_{n\to\infty} \left(\tfrac{1}{2} B h\right)$$

total base ↗ ↖ height of triangle

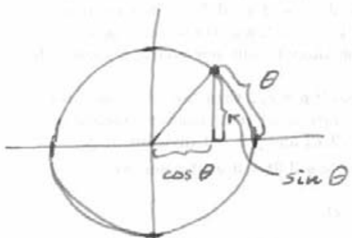
$$= \tfrac{1}{2}(2\pi r)(r)$$

length of an unwrapped circle ↗ ↖ radius of the circle

$$= \pi r^2.$$

So the area of a circle is πr^2 if the circle is radius r.

<u>Trigonometric functions</u>



$\sin\theta$ is the y-coordinate of a point at distance θ on a circle of radius 1

$\cos\theta$ is the x-coordinate of a point at distance θ on a circle of radius 1.

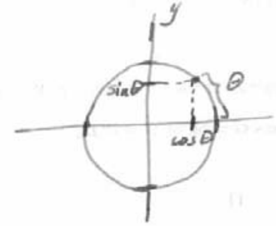
$$\tan\theta = \frac{\sin\theta}{\cos\theta},\quad \cot\theta = \frac{\cos\theta}{\sin\theta},\quad \sec\theta = \frac{1}{\cos\theta},\quad \csc\theta = \frac{1}{\sin\theta}$$

Since the equation of a circle of radius 1 is ④

$x^2+y^2=1$ this forces

$$\sin^2\theta + \cos^2\theta = 1.$$

The pictures



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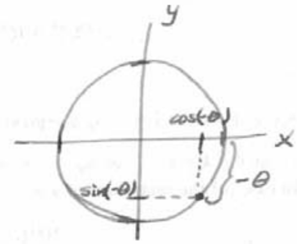
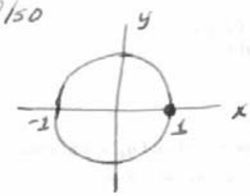


show that

$$\sin(-\theta) = -\sin\theta \quad\text{and}\quad \cos(-\theta) = \cos\theta$$

Also



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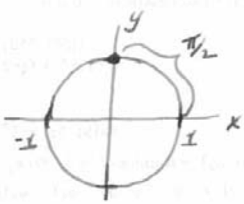
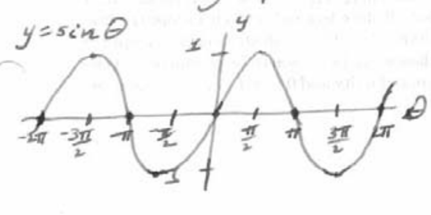


show that

$\sin 0 = 0$, $\cos 0 = 1$ and $\sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$

Draw the graphs $y=\sin\theta$ and $y=\cos\theta$



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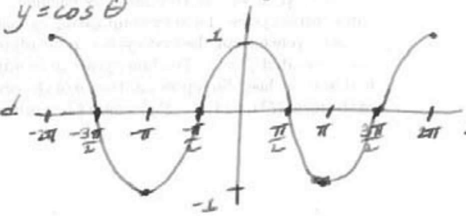

by seeing how the x and y coordinates change as you walk around the circle.

Example Verify $\frac{\sec B}{\cos B} - \frac{\tan B}{\cot B} = 1$.

$$\frac{\sec B}{\cos B} - \frac{\tan B}{\cot B} = \frac{\frac{1}{\cos B}}{\cos B} - \frac{\frac{\sin B}{\cos B}}{\frac{\cos B}{\sin B}} = \frac{1}{\cos^2 B} - \frac{\sin^2 B}{\cos^2 B}$$

$$= \frac{1-\sin^2 B}{\cos^2 B} = \frac{\cos^2 B}{\cos^2 B} = \underline{1}$$

Example Verify $\cot\alpha - \cot\beta = \frac{\sin(\beta-\alpha)}{\sin\alpha\sin\beta}$

Lefthand side $= \cot\alpha - \cot\beta = \frac{\cos\alpha}{\sin\alpha} - \frac{\cos\beta}{\sin\beta}$

$$= \frac{\cos\alpha\sin\beta - \cos\beta\sin\alpha}{\sin\alpha\sin\beta}$$

Right Hand Side $= \frac{\sin(\beta-\alpha)}{\sin\alpha\sin\beta} = \frac{\sin\beta\cos(-\alpha) + \cos\beta\sin(-\alpha)}{\sin\alpha\sin\beta}$

$$= \frac{\sin\beta\cos\alpha + \cos\beta(-\sin\alpha)}{\sin\alpha\sin\beta} = \frac{\sin\beta\cos\alpha - \cos\beta\sin\alpha}{\sin\alpha\sin\beta}$$

So

Left Hand Side = Right Hand Side.

Example Verify $\frac{\tan A - \sin A}{\sec A} = \frac{\sin^3 A}{1+\cos A}$

$$\frac{\tan A - \sin A}{\sec A} \stackrel{?}{=} \frac{\sin^3 A}{1+\cos A}$$

So $(1+\cos A)(\tan A - \sin A) \stackrel{?}{=} \sin^3 A \sec A$

So

$\tan A + \cos A\tan A - \sin A - \sin A\cos A \stackrel{?}{=} \sin^3 A\sec A$

So $\frac{\sin A}{\cos A} + \cos A\frac{\sin A}{\cos A} - \sin A - \sin A\cos A \stackrel{?}{=} \sin^3 A\frac{1}{\cos A}$

So $\frac{\sin A}{\cos A} + \sin A - \sin A - \sin A\cos A \stackrel{?}{=} \frac{\sin^3 A}{\cos A}$

So $\frac{\sin A - \sin A\cos^2 A}{\cos A} \stackrel{?}{=} \frac{\sin^3 A}{\cos A}$

So $\sin A - \sin A\cos^2 A \stackrel{?}{=} \sin^3 A$

So $1 - \cos^2 A \stackrel{?}{=} \sin^2 A$

YES because $\sin^2 A + \cos^2 A = 1$.