MATH 221 Lecture 20, October 25, 2000

Optimization

Critical points are where maxima and minima might occur.

Example Find the local maxima and minima of $f(x) = 2x^3 - 24x + 107$ in the interval $[1,3]$.

The critical points are

(a) points where $\frac{df}{dx}$ is 0,

(b) points where $f(x)$ is not continuous or not differentiable

(c) points on the boundary of where $f(x)$ is defined

For $f(x) = 2x^3 - 24x + 107$ in the interval $[1,3]$,

$x=1$ and $x=3$ are critical points of type (c),

and

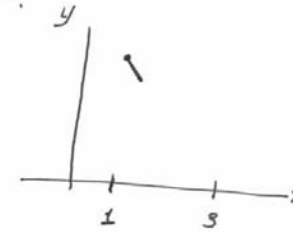
$\frac{df}{dx} = 6x^2 - 24$ and $6x^2 - 24 = 0$ when $x^2 = \frac{24}{6} = 4$

So $x = \pm 2$ is when $\frac{df}{dx}$ is 0.

So $x=2$ is a critical point in $[1,3]$.

Critical point $x=1$:

$$\left.\frac{df}{dx}\right|_{x=1} = \left.6x^2 - 24\right|_{x=1} = 6 - 24 < 0.$$

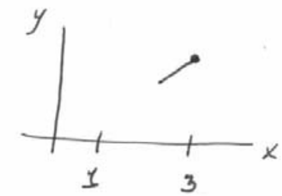


So $f(x)$ is decreasing at $x=1$

So (from the picture) $x=1$ is a maximum

Critical point $x=3$:

$$\left.\frac{df}{dx}\right|_{x=3} = \left.6x^2 - 24\right|_{x=3} = 6\cdot 3^2 - 24 = 30 > 0.$$



So $f(x)$ is increasing at $x=3$.

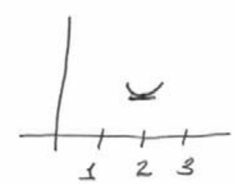
So (from the picture) $x=3$ is a maximum

Critical point $x=2$:

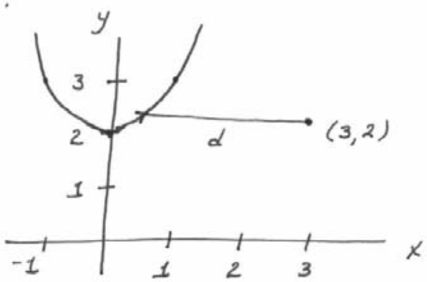
$$\left.\frac{df}{dx}\right|_{x=2} = 0. \quad \left.\frac{d^2f}{dx^2}\right|_{x=2} = \left.12x\right|_{x=2} = 24 > 0.$$

So $f(x)$ is flat and concave up at $x=2$.

So $x=2$ is a minimum



<u>Example</u> An enemy jet is flying along the curve $y = x^2+2$. A soldier is placed at the point $(3,2)$. At what point will the jet be at when the soldier and the jet will be the closest?



If the jet is at the point (p,q) the distance between them is

$$d = \sqrt{(p-3)^2 + (q-2)^2}$$

The point (p,q) is on the curve $y = x^2+2$ so

$$q = p^2+2.$$

So $d = \sqrt{(p-3)^2 + (p^2+2-2)^2}$

We want to minimize d (as the jet moves, i.e. as p changes). The distance d will be minimum at the same time that d^2 will be minimum.

So we can minimize d^2.

$$d^2 = (p-3)^2 + (p^2)^2 = (p-3)^2 + p^4.$$

Find a critical point. When is

$$\frac{d\,d^2}{dp} = 2(p-3) + 4p^3 = 4p^3 + 2p - 6$$
$$= (p-1)(4p^2+4p+6)$$

equal to 0?? When $p=1$.

So $\left.\frac{d\,d^2}{dp}\right|_{p=1} = 0$, so $p=1$ is a critical point.

From the picture we can confirm that when the jet is at $(1,3)$ (i.e. $p=1$, $q=3$) the distance to the soldier is minimum.

<u>Example</u> Maximize the volume of a cone with a given slant height. Show that the angle of inclination is $\tan^{-1}\sqrt{2}$.

l = slant height

θ = angle of inclination

$\frac{r}{l} = \sin\theta \qquad \frac{h}{l} = \cos\theta$

Volume of a cone is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (l\sin\theta)^2 h = \frac{1}{3}\pi (l\sin\theta)^2 l\cos\theta$$

l is fixed (given slant height).

We want to maximize V as θ changes

$$\frac{dV}{d\theta} = \frac{d\ \frac{1}{3}\pi l^3 \sin^2\theta \cos\theta}{d\theta}$$

$$= \frac{1}{3}\pi l^3 \left(2\sin\theta\cos^2\theta - \sin^2\theta \sin\theta\right)$$

$$= \frac{1}{3}\pi l^3 \sin\theta \left(2\cos^2\theta - \sin^2\theta\right)$$

A critical point is when $\frac{dV}{d\theta}$ is zero or

when $2\cos^2\theta - \sin^2\theta = 0$ or $\sin\theta = 0$.

So $2 = \tan^2\theta$ or $\theta = 0$.

So $\sqrt{2} = \tan\theta$ or $\theta = 0$.

So $\theta = \tan^{-1}\sqrt{2}$ or $\theta = 0$.

When $\theta = 0$ the cone looks like / which clearly does not have maximum volume.

So $\theta = \tan^{-1}\sqrt{2}$ maximizes volume.