MATH 221 Lecture 24, November 4, 2000. ①

Example $\int \frac{\sin x}{\sin x - \cos x}\, dx = \int \frac{\sin x - \cos x + \sin x + \cos x}{\sin x - \cos x} \cdot \frac{1}{2}\, dx$

$= \frac{1}{2}\int \frac{\sin x - \cos x}{\sin x - \cos x} + \frac{\sin x + \cos x}{\sin x - \cos x}\, dx$

$= \frac{1}{2}\int \left(1 + \frac{\sin x + \cos x}{\sin x - \cos x}\right) dx = \frac{1}{2}\left(x + \ln(\sin x - \cos x)\right) + c$

since

$$\frac{d\, \frac{1}{2}(x + \ln(\sin x - \cos x))}{dx} = \frac{1}{2}\left(1 + \frac{1}{\sin x - \cos x}(\cos x + \sin x)\right).$$

Example $\int \frac{x^3}{1+x^8}\, dx = \int \frac{x^3}{1+(x^4)^2}\, dx$

$= \int \frac{1}{4} \cdot \frac{4x^3}{1+(x^4)^2}\, dx = \frac{1}{4} \tan^{-1}(x^4) + c$

since $\frac{d\, \frac{1}{4}\tan^{-1}x^4}{dx} = \frac{1}{4}\, \frac{1}{1+(x^4)^2}\, \frac{dx^4}{dx} = \frac{1}{4}\, \frac{4x^3}{1+(x^4)^2}$

Example $\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right) dx = \int \tan^{-1}\left(\frac{2\sin x\cos x}{1+\cos^2 x - \sin^2 x}\right) dx$

②

$= \int \tan^{-1}\left(\frac{2\sin x \cos x}{\cos^2 x + \cos^2 x}\right) dx = \int \tan^{-1}\left(\frac{2\sin x\cos x}{2\cos^2 x}\right) dx$

$= \int \tan^{-1}\left(\frac{\sin x}{\cos x}\right) dx = \int \tan^{-1}(\tan x)\, dx = \int x\, dx$

$= \frac{x^2}{2} + c.$

Example $\int \cos^{-1}(\sin x)\, dx$ Let $x = \sin^{-1} y$.

Then $\frac{dx}{dy} = \frac{1}{\sqrt{1-y^2}}$.

$\int \cos^{-1}(\sin x)\, dx = \int \cos^{-1}(\sin x) \frac{dx}{dy}\, dy$

$= \int \cos^{-1}(\sin(\sin^{-1} y)) \frac{1}{\sqrt{1-y^2}}\, dy$

$= \int \cos^{-1} y \, \frac{1}{\sqrt{1-y^2}}\, dy.$

$= -\int \cos^{-1} y \, \frac{-1}{\sqrt{1-y^2}}\, dy = -\frac{(\cos^{-1} y)^2}{2} + c$

$= -\frac{(\cos^{-1}(\sin x))^2}{2} + c.$

Another way:

$$\cos^{-1}(\sin x) = y$$

$$\sin x = \cos y.$$

So $y = \frac{\pi}{2} - x$.

So $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$.

So $\int \cos^{-1}(\sin x)\,dx = \int \left(\frac{\pi}{2} - x\right) dx = \frac{\pi}{2}x - \frac{x^2}{2} + c.$

<u>Example</u> $\int \frac{2x^2+x-2}{x-2}\,dx = \int \frac{2x(x-2)+5x-2}{x-2}\,dx$

$= \int 2x + \frac{5x-2}{x-2}\,dx = \int 2x + \frac{5(x-2)+8}{x-2}\,dx.$

$= \int 2x + 5 + \frac{8}{x-2}\,dx = x^2 + 5x + 8\ln(x-2) + c.$

<u>Definite integrals</u>

Warning: The following is not quite correct, though it is correct most of the time

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a).$$

<u>Example</u> $\int_1^2 x^{-2}\,dx = \left.\frac{x^{-1}}{-1} + c\,\right|_{1=x}^{2=x} = (-2^{-1}+c) - (-1^{-1}+c)$

$= \frac{-1}{2} + c + 1 - c = \frac{1}{2}.$

<u>Example</u> $\int_0^4 \sqrt{x}\,dx = \int_0^4 x^{\frac{1}{2}}\,dx = \left.\frac{2}{3}x^{3/2} + c\,\right|_{0=x}^{4=x}$

$= \frac{2}{3}4^{3/2} + c - \left(\frac{2}{3}0^{3/2} + c\right) = \frac{2}{3}2^3 + c - c = \frac{2}{3}\cdot 8 = \frac{16}{3}$

<u>Example</u> $\int_1^3 \left(\frac{1}{t^2} - \frac{1}{t^4}\right) dt = \int_1^3 (t^{-2} - t^{-4})\,dt$

$= \left.\frac{t^{-1}}{-1} - \frac{t^{-3}}{-3} + c\,\right|_{t=1}^{t=3} = \left(\frac{3^{-1}}{-1} - \frac{3^{-3}}{-3} + c\right) - \left(\frac{1^{-1}}{-1} - \frac{1^{-3}}{-3} + c\right)$

$= -\frac{1}{3} + \frac{1}{3^4} + c + 1 - \frac{1}{3} - c = 1 + \frac{1}{81} - \frac{2}{3} = \frac{1}{3} + \frac{1}{81} = \frac{28}{81}.$

<u>Example</u> $\int_{-3}^0 (5y^4 - 6y^2 + 14)\,dy = \left.(y^5 - 2y^3 + 14y + c)\right|_{y=-3}^{y=0}$

$= (0 - 0 + 0 + c) - ((-3)^5 - 2(-3)^3 + 14(-3) + c)$

$= c + 3^5 - 2\cdot 3^3 + 14\cdot 3 - c = 243 - 54 + 42 = 231.$

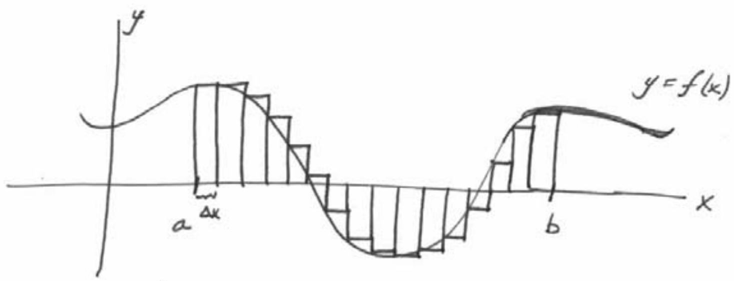
BUT

$\int_a^b \frac{df}{dx}\,dx$ is not always equal to $f(b) - f(a)$.

What does $\int_a^b f(x)\,dx$ really mean??

$\int_a^b f(x)\,dx$ really is

$$\lim_{\Delta x \to 0} \left(f(a)\Delta x + f(a+\Delta x)\Delta x + \cdots + f(b-2\Delta x)\Delta x + f(b-\Delta x)\Delta x \right)$$



Think of $\int_a^b f(x)\,dx$ as saying

Add up the areas $f(x)dx$ from a to b

where

$f(x)dx$ is the "area" of an infinitesimally small box with height $f(x)$ and width dx