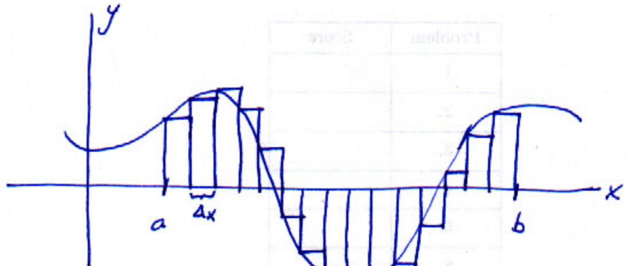
MATH 221 Lecture 25, November 6, 2000

①

$\int_a^b f(x)\,dx$ really means

$$\lim_{\Delta x \to 0} \left(f(a)\Delta x + f(a+\Delta x)\Delta x + \cdots + f(b-\Delta x)\Delta x\right)$$

$= \lim_{\Delta x \to 0}$ (add up the areas of little boxes $f(a+k\Delta x)$, Δx)



The first box $f(a)$, Δx has area $f(a)\Delta x$

The second box $f(a+\Delta x)$, Δx has area $f(a+\Delta x)\Delta x$

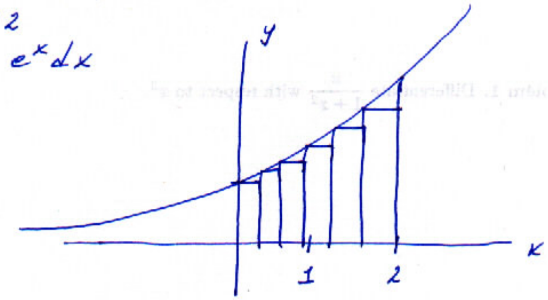
So think of $\int_a^b f(x)\,dx$ as

adding up areas from a to b of

infinitesimally small boxes $f(x)$, dx with area $f(x)\,dx$.

Example $\int_0^2 e^x\,dx$

②



Suppose $\Delta x = \frac{1}{3}$

$$e^0\Delta x + e^{\Delta x}\Delta x + e^{2\Delta x}\Delta x + e^{3\Delta x}\Delta x + e^{4\Delta x}\Delta x + \cdots + e^{2-\Delta x}\Delta x$$

$$= e^0\tfrac{1}{3} + e^{1/3}\tfrac{1}{3} + e^{2/3}\tfrac{1}{3} + e^{3/3}\tfrac{1}{3} + e^{4/3}\tfrac{1}{3} + e^{5/3}\tfrac{1}{3}$$

$$= \tfrac{1}{3}\left(1 + e^{1/3} + (e^{1/3})^2 + (e^{1/3})^3 + (e^{1/3})^4 + (e^{1/3})^5\right)$$

$$= \tfrac{1}{3}\left(\frac{(e^{1/3})^6 - 1}{e^{1/3}-1}\right) = \tfrac{1}{3}\left(\frac{e^{6/3}-1}{e^{1/3}-1}\right) = (e^2-1)\left(\frac{\frac{1}{3}}{e^{1/3}-1}\right)$$

Suppose $\Delta x = \frac{1}{5}$

$$e^0\Delta x + e^{\Delta x}\Delta x + e^{2\Delta x}\Delta x + \cdots + e^{2-\Delta x}\Delta x$$

$$= e^0\tfrac{1}{5} + e^{1/5}\tfrac{1}{5} + e^{2/5}\tfrac{1}{5} + e^{3/5}\tfrac{1}{5} + \cdots + e^{9/5}\tfrac{1}{5}$$

$$= \tfrac{1}{5}\left(e^0 + e^{1/5} + e^{2/5} + e^{3/5} + e^{4/5} + \cdots + e^{9/5}\right)$$

$$= \tfrac{1}{5}\left(e^0 + e^{1/5} + (e^{1/5})^2 + (e^{1/5})^3 + \cdots + (e^{1/5})^9\right)$$

$$= \tfrac{1}{5}\left(\frac{(e^{1/5})^{10} - 1}{e^{1/5}-1}\right) = (e^2-1)\left(\frac{\frac{1}{5}}{e^{1/5}-1}\right)$$

Suppose $\Delta x = \frac{1}{N}$.

$$e^0\Delta x + e^{\Delta x}\Delta x + e^{2\Delta x}\Delta x + e^{3\Delta x}\Delta x + \cdots + e^{2-\Delta x}\Delta x$$

$$= e^0\frac{1}{N} + e^{1/N}\frac{1}{N} + e^{2/N}\frac{1}{N} + e^{3/N}\frac{1}{N} + e^{4/N}\frac{1}{N} + \cdots + e^{2-\frac{1}{N}}\frac{1}{N}$$

$$= (e^2-1)\left(\frac{\frac{1}{N}}{e^{1/N}-1}\right)$$

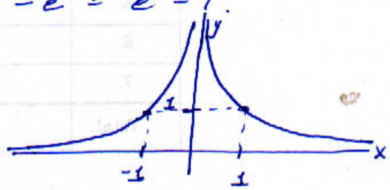
So $\displaystyle\lim_{\Delta x\to 0}\left(e^0\Delta x + e^{\Delta x}\Delta x + \cdots + e^{2-\Delta x}\Delta x\right)$

$$= \lim_{\Delta x\to 0}(e^2-1)\frac{\Delta x}{(e^{\Delta x}-1)} = (e^2-1)\cdot 1 = e^2-1.$$

Note: $\displaystyle\int_0^2 e^x\,dx = e^x + c\Big|_{x=0}^{x=2} = (e^2+c)-(e^0+c)$

$$= e^2+c-e^0-c = e^2-e^0 = e^2-1.$$

Example $\displaystyle\int_{-1}^{1}\frac{1}{x^2}\,dx$



By adding up little boxes:

$$\int_{-1}^{1}\frac{1}{x^2}\,dx = \lim_{\Delta x\to 0}\left(\frac{1}{(-1)^2}\Delta x + \frac{1}{(-1+\Delta x)^2}\Delta x + \frac{1}{(-1+2\Delta x)^2}\Delta x + \cdots + \frac{1}{(1-\Delta x)^2}\Delta x\right)$$

$$= \lim_{\Delta x\to 0}\left(1\cdot\Delta x + \frac{1}{(-1+\Delta x)^2}\Delta x + \cdots + \frac{1}{0^2}\Delta x + \cdots + \frac{1}{(1-\Delta x)^2}\Delta x\right)$$

OOPS!!

So $\displaystyle\int_{-1}^{1}\frac{1}{x^2}\,dx$ is UNDEFINED.

Note:

$$\int_{-1}^{1}\frac{1}{x^2}\,dx = \int_{-1}^{1}x^{-2}\,dx = \frac{x^{-1}}{-1}+c\Big|_{x=-1}^{x=1}$$

$$= \left(\frac{1^{-1}}{-1}+c\right) - \left(\frac{(-1)^{-1}}{-1}+c\right) = -1+c-1-c = -2$$

So this is a case where

$$\int_a^b \frac{df}{dx}\,dx \neq f(b)-f(a)$$

i.e. adding up areas of little boxes

and doing the indefinite integral and plugging in

give different answers.

The fundamental theorem of calculus says

$$\int_a^b \frac{df}{dx}\,dx = f(b)-f(a)$$

(is not a lie) provided $f(x)$ doesn't do anything bad between a and b. It should be

(a) defined everywhere between a and b,

(b) continuous everywhere between a and b,

(c) differentiable everywhere between a and b.

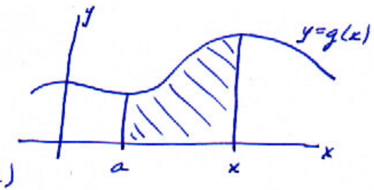
The fundamental theorem of calculus says

$$\begin{array}{l}\text{Area under } g(x) \\ \text{from } a \text{ to } b\end{array} = A(b) - A(a)$$

where $\int g(x)\,dx = A(x) + c$.

Why does this work?

Let $A(x) =$ area under $g(x)$ from a to x.



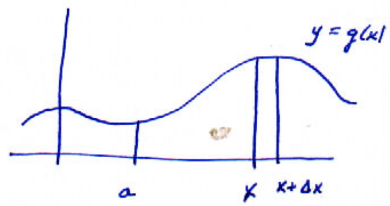
Then

$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\text{area of last little box})}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{g(x)\,\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} g(x) = g(x).$$



So $\int g(x)\,dx = A(x) + c$.

So $A(b) - A(a) = \begin{pmatrix}\text{area under } g(x) \\ \text{from } a \text{ to } b\end{pmatrix} - \begin{pmatrix}\text{area under } g(x) \\ \text{from } a \text{ to } a\end{pmatrix}$

$$= \begin{array}{l}\text{area under } g(x) \\ \text{from } a \text{ to } b.\end{array}$$