MATH 221, Lecture 26, November 8, 2000

Computing Areas and Volumes

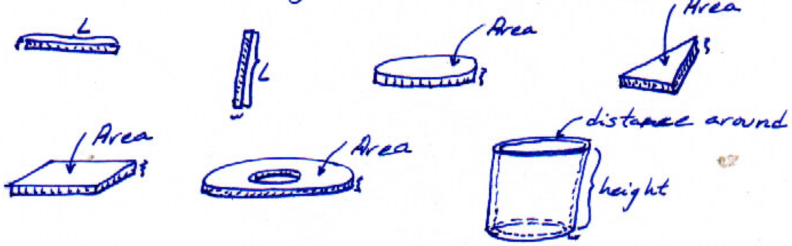
(1) Carefully draw the region.

(2) Slice it up; draw a typical slice.

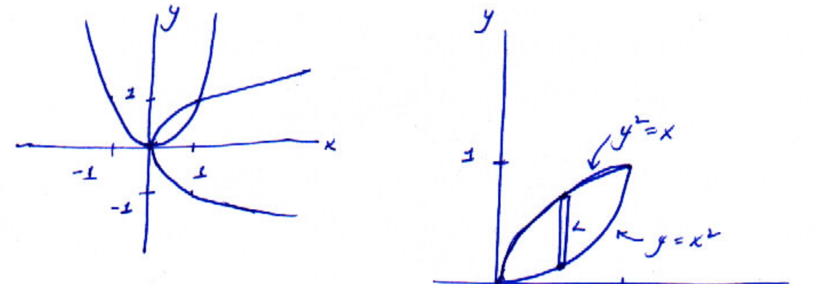
(3) Find the volume of a slice.

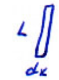
(4) Add up the volumes of the slices with an integral.

Typical slices might look like:



Example Calculate the area of the region bounded by the parabolas $y=x^2$ and $y^2=x$.



Slice: 

Area of Slice: $L\,dx$

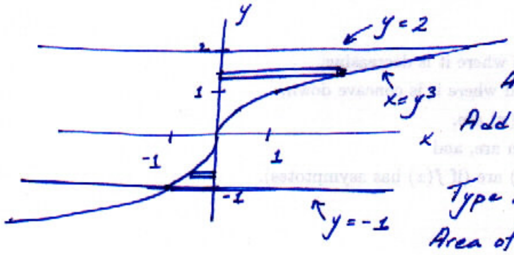
Add slices from $x=0$ to $x=1$.

$$\int_{x=0}^{x=1} L\,dx = \int_{x=0}^{x=1} (y_{upper} - y_{lower})\,dx = \int_{x=0}^{x=1} (\sqrt{x} - x^2)\,dx$$

$$= \frac{2}{3}x^{3/2} - \frac{x^3}{3}\Big|_{x=0}^{x=1} = \left(\frac{2}{3}1^{3/2} - \frac{1}{3}\right) - \left(\frac{2}{3}0^{3/2} - \frac{0^3}{3}\right)$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

Example Find the area of the region bounded by $y=-1$, $y=2$, $x=y^3$ and $x=0$.



Type 1 slice: L_1 dy

Area of slice: $L_1\,dy$

Add slices from $y=0$ to $y=2$.

Type 2 slice: dy L_2

Area of slice: $L_2\,dy$

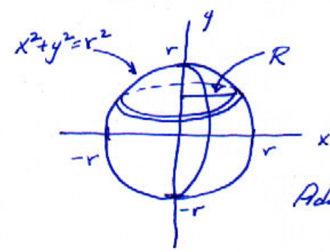
Add slices from $y=-1$ to $y=0$.

$$\int_{y=0}^{y=2} L_1\,dy + \int_{y=-1}^{y=0} L_2\,dy = \int_{y=0}^{y=2} x\,dy + \int_{y=-1}^{y=0} (-x)\,dy$$

$$= \int_{y=0}^{y=2} y^3\,dy + \int_{y=-1}^{y=0} -y^3\,dy = \frac{y^4}{4}\Bigg|_{y=0}^{y=2} + \frac{-y^4}{4}\Bigg|_{y=-1}^{y=0}$$

$$= \left(\frac{2^4}{4} - \frac{0}{4}\right) + \left(-\frac{0^4}{4} - \left(\frac{-(-1)^4}{4}\right)\right) = 2^2 + \frac{1}{4} = 4\tfrac{1}{4}.$$

Example Find the volume of a sphere of radius r.



Slice:

Volume of slice: $\pi R^2\,dy$

Add slices from $y=-r$ to $y=r$

$$\text{Volume of sphere} = \int_{y=-r}^{y=r} \pi R^2\,dy = \int_{y=-r}^{y=r} \pi x^2\,dy$$

$$= \int_{y=-r}^{y=r} \pi(r^2-y^2)\,dy = \pi\left(r^2y - \frac{y^3}{3}\right)\Bigg|_{y=-r}^{y=r}$$

$$= \pi\left(r^2\cdot r - \frac{r^3}{3}\right) - \pi\left(r^2(-r) - \frac{(-r)^3}{3}\right)$$

$$= \pi\,\frac{2}{3}r^3 + \pi r^3 - \frac{\pi r^3}{3} = \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3$$

$$= \frac{4}{3}\pi r^3.$$

Example Compute $\int_{-a}^{a} \sqrt{a^2-x^2}\,dx$.

If $x = a\sin\theta$,

then

$$\int_{-a}^{a} \sqrt{a^2-x^2}\,dx$$

$$= \int_{-a}^{a} \sqrt{a^2-a^2\sin^2\theta}\,dx$$

$$= \int_{x=-a}^{x=a} \sqrt{a^2(1-\sin^2\theta)}\,dx$$

$$= \int_{x=-a}^{x=a} \sqrt{a^2\cos^2\theta}\,dx$$

$$= \int_{x=-a}^{x=a} a\cos\theta\,dx$$

$$= \int_{x=-a}^{x=a} a\cos\theta\,\frac{dx}{d\theta}\,d\theta$$

$$= \int_{x=-a}^{x=a} a\cos\theta\, a\cos\theta\,d\theta$$

$$= \int_{x=-a}^{x=a} a^2\cos^2\theta\,d\theta$$

$$= \int_{x=-a}^{x=a} \tfrac{1}{2}a^2(\cos^2\theta+\cos^2\theta)\,d\theta$$

$$= \int_{x=-a}^{x=a} \tfrac{1}{2}a^2(\cos^2\theta+1-\sin^2\theta)\,d\theta$$

$$= \int_{x=-a}^{x=a} \tfrac{1}{2}a^2(\cos^2\theta-\sin^2\theta+1)\,d\theta$$

$$= \int_{x=-a}^{x=a} \tfrac{1}{2}a^2(\cos 2\theta+1)\,d\theta$$

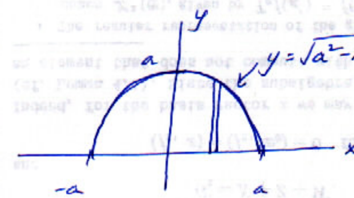
$$= \tfrac{1}{2}a^2\left(\frac{\sin 2\theta}{2}+\theta\right)\Bigg|_{x=-a}^{x=a}$$

$$= \tfrac{1}{2}a^2\left(\frac{\sin 2\theta}{2}+\theta\right)\Bigg|_{\sin\theta=-1}^{\sin\theta=1}$$

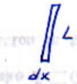
$$= \tfrac{1}{2}a^2\left(\frac{\sin 2\theta}{2}+\theta\right)\Bigg|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}$$

$$= \tfrac{1}{2}a^2\left(\frac{\sin\pi}{2}+\frac{\pi}{2}\right) - \tfrac{1}{2}a^2\left(\frac{\sin(-\pi)}{2}-\frac{\pi}{2}\right) = \tfrac{1}{2}a^2\frac{\pi}{2} - \tfrac{1}{2}a^2\left(\frac{-\pi}{2}\right) = \frac{\pi a^2}{4}.$$

Example Compute $\int_{x=-a}^{x=a} \sqrt{a^2-x^2}\,dx$



Slice:



Area of slice: $L\,dx$

Add slices from $x=-a$ to $x=a$.

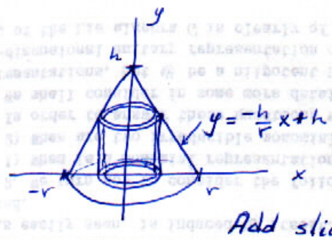
$$\frac{\pi a^2}{2} = \text{Area of semicircle} = \int_{x=-a}^{x=a} L\,dx$$

$$= \int_{x=-a}^{x=a} y\,dx$$

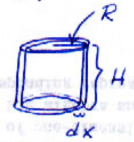
$$= \int_{x=-a}^{x=a} \sqrt{a^2-x^2}\,dx.$$

So $\frac{\pi a^2}{2} = \int_{x=-a}^{x=a} \sqrt{a^2-x^2}\,dx$.

Example: Find the volume of a right circular cone of height h and radius r.



Slice:



Volume of Slice: $2\pi R H\,dx$

Add slices from $x=0$ to $x=r$

$$\int_{x=0}^{x=r} 2\pi R H\,dx = \int_{x=0}^{x=r} 2\pi x y\,dx = \int_{x=0}^{x=r} 2\pi x\left(-\frac{h}{r}x+h\right)dx$$

$$= \int_{x=0}^{x=r}\left(-\frac{2\pi h}{r}x^2 + 2\pi h x\right)dx$$

$$= -\frac{2\pi h}{r}\frac{x^3}{3} + \pi h x^2 \Big|_{x=0}^{x=r} = \left(-\frac{2\pi h}{r}\frac{r^3}{3} + \pi h r^2\right) - (-0+0)$$

$$= -\frac{2}{3}\pi r^2 h + \pi r^2 h = \frac{1}{3}\pi r^2 h$$