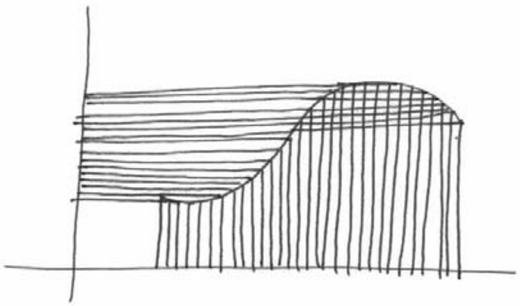
MATH 221, Lecture 32, November 27, 2000 ①

Lengths of curves

Idea: Use the grid to slice up the curve into little pieces.

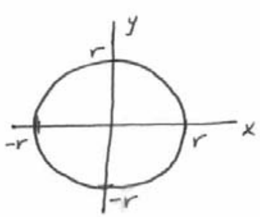


Each little piece $dy \,\Delta\, ds$ (dx) has length

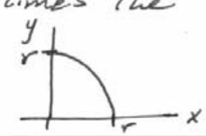
$$ds = \sqrt{(dx)^2 + (dy)^2}.$$

Add up the lengths of the little pieces with an integral.

Example Use integration to find the length of a circle of radius r.



The length of the whole circle is 4 times the length of



Divide this part of the curve into little ②

pieces $dy \,\Delta\, ds$ (dx). Each little piece has length

$ds = \sqrt{(dx)^2 + (dy)^2}$. Add up the lengths of the little pieces with an integral.

$$\int_{x=0}^{x=r} ds = \int_{x=0}^{x=r} \sqrt{(dx)^2 + (dy)^2} = \int_{x=0}^{x=r} \frac{\sqrt{(dx)^2 + (dy)^2}}{dx}\, dx$$

$$= \int_{x=0}^{x=r} \sqrt{\frac{(dx)^2 + (dy)^2}{(dx)^2}}\, dx = \int_{x=0}^{x=r} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{x=0} \sqrt{1 + \left(\frac{-2x}{2y}\right)^2}\, dx,$$ since for $x^2 + y^2 = r^2$

$$2x + 2y\frac{dy}{dx} = 0,$$

and so $\dfrac{dy}{dx} = \dfrac{-2x}{2y}$.

So $$\int_{x=0}^{x=r} ds = \int_{x=0}^{x=r} \sqrt{1 + \frac{x^2}{y^2}}\, dx = \int_{x=0}^{x=r} \sqrt{\frac{y^2 + x^2}{y^2}}\, dx = \int_{x=0}^{x=r} \sqrt{\frac{r^2}{r^2 - x^2}}\, dx$$

$$= \int_{x=0}^{x=r} r\frac{1}{\sqrt{r^2 - x^2}}\, dx = \int_{x=0}^{x=r} r\frac{1}{\sqrt{r^2\left(1 - \frac{x^2}{r^2}\right)}}\, dx = \int_{x=0}^{x=r} \frac{1}{\sqrt{1 - \left(\frac{x}{r}\right)^2}}\, dx$$

$$= \int_{x=0}^{x=r} \frac{r\cdot\frac{1}{r}}{\sqrt{1-(\frac{x}{r})^2}}\,dx = r\sin^{-1}\left(\frac{x}{r}\right)\Big|_{x=0}^{x=r}$$

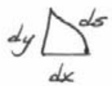
$$= r\sin^{-1}1 - r\sin^{-1}(0) = r\frac{\pi}{2} - 0.$$

So the total length of the circle is

$$4\left(\frac{r\pi}{2}\right) = 2\pi r.$$

Example Find the length of the curve

$x = t - \sin t$, $y = 1-\cos t$, $0 \le t \le 2\pi$.

Divide the curve into little pieces 

Each little piece has length $ds = \sqrt{(dx)^2+(dy)^2}$

Add up the lengths of the little pieces:

$$\int_{t=0}^{t=2\pi} ds = \int_{t=0}^{t=2\pi} \sqrt{(dx)^2+(dy)^2} = \int_{t=0}^{t=2\pi} \frac{\sqrt{(dx)^2+(dy)^2}}{dt}\,dt$$

$$= \int_{t=0}^{t=2\pi} \sqrt{\frac{(dx)^2+(dy)^2}{(dt)^2}}\,dt = \int_{t=0}^{t=2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_{t=0}^{t=2\pi} \sqrt{(1-\cos t)^2+(\sin t)^2}\,dt$$

$$= \int_{t=0}^{t=2\pi} \sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\,dt$$

$$= \int_{t=0}^{t=2\pi} \sqrt{1-2\cos t+1}\,dt = \int_{t=0}^{t=2\pi} \sqrt{2-2\cos t}\,dt$$

$$= \int_{t=0}^{t=2\pi} \sqrt{2-2\cos\left(\frac{t}{2}+\frac{t}{2}\right)}\,dt = \int_{t=0}^{t=2\pi} \sqrt{2-2\left(\cos^2\left(\frac{t}{2}\right)-\sin^2\left(\frac{t}{2}\right)\right)}\,dt$$

$$= \int_{t=0}^{t=2\pi} \sqrt{2}\sqrt{1-\cos^2\left(\frac{t}{2}\right)+\sin^2\left(\frac{t}{2}\right)}\,dt = \int_{t=0}^{t=2\pi} \sqrt{2}\sqrt{\sin^2\left(\frac{t}{2}\right)+\sin^2\left(\frac{t}{2}\right)}\,dt$$

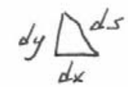
$$= \int_{t=0}^{t=2\pi} \sqrt{2}\sqrt{2\sin^2\left(\frac{t}{2}\right)}\,dt = \int_{t=0}^{t=2\pi} \sqrt{2}\sqrt{2}\sin\left(\frac{t}{2}\right)dt$$

$$= 2\left(-\cos\left(\frac{t}{2}\right)\right)\cdot 2\Big|_{t=0}^{t=2\pi} = -4\cos\left(\frac{2\pi}{2}\right) - (-4\cos 0)$$

$$= (-4)(-1) + 4\cdot 1 = 4+4 = 8.$$

Example Find the length of the curve

$x = \frac{3}{5}y^{5/3} - \frac{3}{4}y^{1/3}$ from $y=0$ to $y=1$.

Divide the curve into little pieces 

Each little piece has length $ds = \sqrt{(dx)^2 + (dy)^2}$

Add up the lengths of the little pieces

$$\int_{y=0}^{y=1} ds = \int_{y=0}^{y=1} \sqrt{(dx)^2 + (dy)^2} = \int_{y=0}^{y=1} \frac{\sqrt{(dx)^2 + (dy)^2}}{dy}\, dy$$

$$= \int_{y=0}^{y=1} \sqrt{\frac{(dx)^2 + (dy)^2}{(dy)^2}}\, dy = \int_{y=0}^{y=1} \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

$$= \int_{y=0}^{y=1} \sqrt{\left(y^{2/3} - \tfrac{1}{4} y^{-2/3}\right)^2 + 1}\, dy = \int_{y=0}^{y=1} \sqrt{y^{4/3} - \tfrac{1}{2} + \tfrac{1}{16} y^{-4/3} + 1}\, dy$$

$$= \int_{y=0}^{y=1} \sqrt{y^{4/3} + \tfrac{1}{2} + \tfrac{1}{16} y^{-4/3}}\, dy = \int_{y=0}^{y=1} \sqrt{\left(y^{2/3} + \tfrac{1}{4} y^{-2/3}\right)^2}\, dy$$

$$= \int_{y=0}^{y=1} \left(y^{2/3} + \tfrac{1}{4} y^{-2/3}\right) dy = \tfrac{3}{5} y^{5/3} + \tfrac{1}{4} \cdot 3\, y^{\frac{1}{3}} \Big|_{y=0}^{y=1}$$

$$= \frac{3}{5} + \frac{3}{4} - (0 + 0) = \frac{12}{20} + \frac{15}{20} = \frac{27}{20}.$$