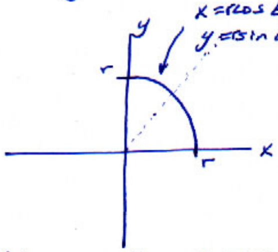
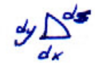
MATH 221 Lecture 34, December 1, 2000 ①

Example Find the center of gravity of the arc length of one quadrant of the circle.



The center of mass will be on the line $y=x$ so its x-coordinate and its y-coordinate will be the same.

Chop up the curve into little pieces 

x-coordinate of position of little piece is x.

Mass of little piece is (density)$\cdot ds$.

Add up little pieces from $\theta=0$ to $\theta=\frac{\pi}{2}$

x-coordinate of center of mass $= \dfrac{\int_{\theta=0}^{\theta=\pi/2} x\,\delta\, ds}{\int_{\theta=0}^{\theta=\pi/2} \delta\, ds}$ where δ = density.

$$= \frac{\int_{\theta=0}^{\theta=\pi/2} x\,\delta\sqrt{(dx)^2+(dy)^2}}{\int_{\theta=0}^{\theta=\pi/2} \delta\sqrt{(dx)^2+(dy)^2}} = \frac{\int_{\theta=0}^{\theta=\pi/2} x\,\delta\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\, d\theta}{\int_{\theta=0}^{\theta=\pi/2} \delta\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\, d\theta}$$

$$= \frac{\int_{\theta=0}^{\theta=\pi/2} \delta r\cos\theta\sqrt{(-r\sin\theta)^2+(r\cos\theta)^2}\, d\theta}{\int_{\theta=0}^{\theta=\pi/2} \delta\sqrt{(-r\sin\theta)^2+(r\cos\theta)^2}\, d\theta}$$

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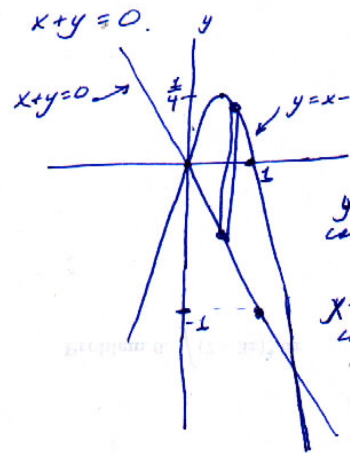
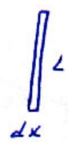
$$= \frac{\int_{\theta=0}^{\theta=\pi/2} \delta r\cos\theta\sqrt{r^2\sin^2\theta+r^2\cos^2\theta}\, d\theta}{\int_{\theta=0}^{\theta=\pi/2} \delta\sqrt{r^2\sin^2\theta+r^2\cos^2\theta}\, d\theta}$$

$$= \frac{\int_{\theta=0}^{\theta=\pi/2} \delta r\cos\theta\, r\, d\theta}{\int_{\theta=0}^{\theta=\pi/2} \delta r\, d\theta} = \frac{\delta r^2\sin\theta\Big|_{\theta=0}^{\theta=\pi/2}}{\delta r\theta\Big|_{\theta=0}^{\theta=\pi/2}}$$

$$= \frac{\delta r^2\sin\frac{\pi}{2} - \delta r^2\cdot 0}{\delta r\frac{\pi}{2} - \delta r\cdot 0} = \frac{\delta r^2}{\delta r\frac{\pi}{2}} = \frac{2r}{\pi}.$$

Example Find the center of gravity of the area bounded by the curve $y = x - x^2$ and the line $x+y=0$.



Slice: 

mass of slice: (density) $L\,dx$

y-coordinate of center of mass of slice is at half way between top and bottom

x-coordinate of center of mass of slice is at x-position of slice

Add up slices from $x=0$ to x at right intersection point.

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When $x+y=0$ and $y=x-x^2$ intersect

$y=-x=x-x^2$. So $x^2-2x=0$.

So $x(x-2)=0$. So $x=0$ or $x=2$.

$$\text{x-coordinate of center of mass of area} = \frac{\int_{x=0}^{x=2} \left(\text{x-position of slice}\right) \delta L \, dx}{\int_{x=0}^{x=2} \delta L \, dx} \quad \text{where } \delta \text{ is density,}$$

$$= \frac{\int_{x=0}^{x=2} x\,\delta(y_{top}-y_{bottom})\,dx}{\int_{x=0}^{x=2} \delta(y_{top}-y_{bottom})\,dx}$$

$$= \frac{\int_{x=0}^{x=2} x\,\delta((x-x^2)-(-x))\,dx}{\int_{x=0}^{x=2} \delta((x-x^2)-(-x))\,dx} = \frac{\int_{x=0}^{x=2} \delta x(2x-x^2)\,dx}{\int_{x=0}^{x=2} \delta(2x-x^2)\,dx}$$

$$= \frac{\int_{x=0}^{x=2} \delta(2x^2-x^3)\,dx}{\int_{x=0}^{x=2} \delta(2x-x^2)\,dx} = \frac{\delta\left(\frac{2x^3}{3}-\frac{x^4}{4}\right)\Big|_{x=0}^{x=2}}{\delta\left(\frac{2x^2}{2}-\frac{x^3}{3}\right)\Big|_{x=0}^{x=2}}$$

$$= \frac{\frac{2\cdot 2^3}{3}-\frac{2^4}{4}-(0-0)}{\frac{2\cdot 2^2}{2}-\frac{2^3}{3}-(0-0)} = \frac{2^4\left(\frac{1}{3}-\frac{1}{4}\right)}{2^3\left(\frac{1}{2}-\frac{1}{3}\right)} = \frac{2\left(\frac{1}{12}\right)}{\left(\frac{1}{6}\right)} = \frac{2}{2} = 1.$$

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$$\text{y-coordinate of center of mass of area} = \frac{\int_{x=0}^{x=2} \left(\text{y-position of slice}\right) \delta L \, dx}{\int_{x=0}^{x=2} \delta L \, dx}$$

$$= \frac{\int_{x=0}^{x=2} \left(\left(\frac{y_{top}-y_{bottom}}{2}\right)+y_{bottom}\right) \delta (y_{top}-y_{bottom})\,dx}{\int_{x=0}^{x=2} \delta(y_{top}-y_{bottom})\,dx}$$

$$= \frac{\int_{x=0}^{x=2} \delta\left(\frac{(x-x^2)-(-x)}{2}+(-x)\right)\left((x-x)^2-(-x)\right)dx}{\int_{x=0}^{x=2} \delta((x-x^2)-(-x))\,dx}$$

$$= \frac{\int_{x=0}^{x=2} \delta\left(-\frac{x^2}{2}\right)(2x-x^2)\,dx}{\int_{x=0}^{x=2} \delta(2x-x^2)\,dx} = \frac{\int_{x=0}^{x=2} \delta\left(-\frac{2x^3}{2}+\frac{x^4}{2}\right)dx}{\int_{x=0}^{x=2} \delta(2x-x^2)\,dx}$$

$$= \frac{\delta\left(-\frac{x^4}{4}+\frac{x^5}{10}\right)\Big|_{x=0}^{x=2}}{\delta\left(x^2-\frac{x^3}{3}\right)\Big|_{x=0}^{x=2}} = \frac{-\frac{2^4}{4}+\frac{2^5}{10}}{2^4-\frac{2^3}{3}} = \frac{2^4\left(-\frac{1}{4}+\frac{1}{5}\right)}{2^3\left(2-\frac{1}{3}\right)}$$

$$= \frac{2\left(\frac{-1}{20}\right)}{\frac{5}{3}} = \frac{-2\cdot 3}{5\cdot 20} = \frac{-3}{5\cdot 10} = \frac{-3}{50}.$$

So the center of mass is $\left(1, \frac{-3}{50}\right)$.