MATH 221 Lecture 35, December 4, 2000 ①

Applications of exponential functions

Example If the bacteria in a culture increase continuously at a rate proportional to the number present, and the initial number is N_0 find the number at time t.

Idea: Change in bacteria is proportional to the amount of bacteria

$$\frac{dB}{dt} = kB.$$

What could B be?

$$\frac{dB}{B} = k\,dt. \quad \text{So} \int \frac{1}{B}\,dB = \int k\,dt.$$

So $\ln B = kt + c$.

So $B = e^{kt+c} = e^c e^{kt} = Ce^{kt}$, where C is a constant.

At time $t=0$, $B = N_0 = Ce^{k\cdot 0} = C$. So $C = N_0$.

So

$$B = N_0 e^{kt}.$$

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Example A roast turkey is taken from an oven when its temperature reaches 185°F and is placed on a table in a room where the temperature is 75°F. It cools at a rate proportional to the difference between its current temperature and the room temperature.

(a) If the temperature of the turkey is 150°F after half an hour what is the temperature after 45 minutes?

(b) When will the turkey have cooled to 100°F?

Idea: change in temperature is proportional to current temperature − room temperature.

$$\frac{dT}{dt} = k(T-R).$$

So $\frac{dT}{T-R} = k\,dt$. So $\int \frac{dT}{T-R} = \int k\,dt$

So $\ln(T-R) = kt + c$

So $T - R = e^{kt+c} = e^c e^{kt} = Ce^{kt}$

where C is a constant.

So $T = Ce^{kt} + R$.

At $t=0$, $T = 185 = Ce^{k\cdot 0} + 75 = C + 75$

So $C = 185 - 75 = 115$.

So $T = 115e^{kt} + 75$.

At $t = \frac{1}{2}$, $T = 115e^{k\frac{1}{2}} + 75 = 150$.

So $e^{k\frac{1}{2}} = \frac{150-75}{115} = \frac{75}{115}$

So $\frac{1}{2}k = \ln\left(\frac{75}{115}\right)$

So $k = 2\ln\left(\frac{75}{115}\right)$.

So $T = 115e^{2\ln\left(\frac{75}{115}\right)t} + 75$.

At $t = \frac{3}{4}$, $T = 115e^{2\ln\left(\frac{75}{115}\right)\frac{3}{4}} + 75 = 115e^{\frac{3}{2}\ln\left(\frac{75}{115}\right)} + 75$

$= 115\left(e^{\ln\left(\frac{75}{115}\right)}\right)^{3/2} + 75 = 115\left(\frac{75}{115}\right)^{3/2} + 75$

If $T = 100$ then $115e^{2\ln\left(\frac{75}{115}\right)t} + 75 = 100$

So $e^{2\ln\left(\frac{75}{115}\right)t} = \frac{100-75}{115} = \frac{25}{115}$

So $2\ln\left(\frac{75}{115}\right)t = \ln\left(\frac{25}{115}\right)$.

So $t = \dfrac{\ln\left(\frac{25}{115}\right)}{2\ln\left(\frac{75}{115}\right)}$.

Example The majority of naturally occurring rhenium is $^{187}_{75}Re$, which is radioactive and has a half life of 7×10^{10} years. In how many years will 5% of the earth's $^{187}_{75}Re$ decompose.

Idea: change in $^{187}_{75}Re$ is proportional to existing amount of $^{187}_{75}Re$.

$\frac{dR}{dt} = kR$.

So $\frac{dR}{R} = kt$. So $\int \frac{dR}{R} = \int kt$.

So $\ln R = kt + C$. So $R = e^{kt+c} = e^{c}e^{kt} = Ce^{kt}$

where C is a constant.

When $t = 0$ the amount is R_0. So $R_0 = Ce^{k\cdot 0} = C$.

When $t = 7\times 10^{10}$ the amount is $\frac{1}{2}R_0$.

So $\frac{1}{2}R_0 = R_0e^{k\cdot 7\cdot 10^{10}}$ So $\frac{1}{2} = e^{k\cdot 7\cdot 10^{10}}$

So $\ln\frac{1}{2} = k\cdot 7\cdot 10^{10}$. So $k = \frac{\ln\left(\frac{1}{2}\right)}{7\times 10^{10}}$.

So $R = R_0e^{\frac{\ln\left(\frac{1}{2}\right)}{7\times 10^{10}}t}$

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We want to know when $R = .05 R_0$.

$$.05 R_0 = R_0 e^{\frac{\ln(1/2)}{7\times10^{10}} t}$$

so $\frac{1}{20} = e^{\frac{\ln(1/2)}{7\times10^{10}} t}$. so $\ln\left(\frac{1}{20}\right) = \frac{\ln(\frac{1}{2})}{7\times10^{10}} t$

so $t = \dfrac{7\times10^{10}\ln(\frac{1}{20})}{\ln(\frac{1}{2})} = \dfrac{7\times10^{10}\ln(\frac{1}{20})}{\ln(\frac{1}{2})}$.

<u>Example</u> If you buy a \$200,000 home and put 10% down and take out a 30 year fixed rate mortgage at 8% per year compute how much your payment would be if you paid it all off in one big payment at the end of 30 years.

Idea: Change in the money is .08 of its current amount.

$$\frac{dM}{dt} = .08 M.$$

so $\frac{dM}{M} = .08dt$. so $\int \frac{dM}{M} = \int .08\,dt$.

so $\ln M = .08t + c$. so $M = e^{.08t+c} = e^c e^{.08t} = Ce^{.08t}$

where C is a constant.

so $M = Ce^{.08t}$.

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At time $t=0$ we owe $200{,}000 - 20{,}000 = 180{,}000$.

$$180{,}000 = Ce^{.08\cdot0} = C.$$

so $M = 180{,}000\, e^{.08t}$.

After 30 years we owe

$$M = 180{,}000\, e^{.08\cdot30} = 180{,}000\, e^{.24} \text{ dollars}.$$

<u>Example</u> If you borrow \$500 on your credit card at 14% interest find the amounts due at the end of 2 years if the interest is compounded

(a) annually
(b) quarterly
(c) monthly
(d) daily
(e) hourly
(f) every second
(g) every nanosecond
(h) continuously.

You owe:

(a) $500 + 500(.14) = 500(1+.14)$ after one year

$500(1+.14)(1+.14)$ after two years.

(b) $500 + 500(\frac{.14}{4}) = 500(1+\frac{.14}{4})$ after one quarter

$500(1+\frac{.14}{4})^2$ after two quarters

$500(1+\frac{.14}{4})^8$ after two years (8 quarters)

(c) $500+500(\frac{.14}{12}) = 500(1+\frac{.14}{12})$ after 1 month.

$500(1+\frac{.14}{12})^{24}$ after two years (24 months).

(d) $500+500(\frac{.14}{365}) = 500(1+\frac{.14}{365})$ after 1 day

$500(1+\frac{.14}{365})^{2\cdot 365}$ after two years ($2\cdot 365$ days)

(e) $500+500(\frac{.14}{365\cdot 12}) = 500(1+\frac{.14}{365\cdot 12})$ after 1 hour.

$500(1+\frac{.14}{365\cdot 12})^{2\cdot 365\cdot 12}$ after two years

(f) $500+500(\frac{.14}{365\cdot 12\cdot 3600}) = 500(1+\frac{.14}{365\cdot 12\cdot 3600})$ after 1 second.

$500(1+\frac{.14}{365\cdot 12\cdot 3600})^{2\cdot 365\cdot 12\cdot 3600}$ after two years

(h) $\lim_{n\to\infty} 500\left(1+\frac{.14}{n}\right)^{n\cdot 2}$

$= \lim_{n\to\infty} 500\left(\left(1+\frac{.14}{n}\right)^n\right)^2 = 500\,(e^{.14})^2 = 500\,e^{.28}$

after two years, since

$\lim_{n\to\infty}\left(1+\frac{.14}{n}\right)^n = \lim_{n\to\infty}\left(e^{\ln(1+\frac{.14}{n})}\right)^n$

$= \lim_{n\to\infty} e^{n\ln(1+\frac{.14}{n})} = \lim_{n\to\infty} e^{n\left(\frac{\ln(1+\frac{.14}{n})}{\frac{.14}{n}}\right)\left(\frac{.14}{n}\right)}$

$= \lim_{n\to\infty} e^{.14\,\frac{\ln(1+\frac{.14}{n})}{.14/n}} = e^{.14\cdot 1} = e^{.14}.$

Example A sample of a wooden artifact from an Egyptian tomb has a $^{14}C/^{12}C$ ratio which is 54.2% of that of freshly cut wood. In approximately what year was the old wood cut? The half life of ^{14}C is 5720 years.

Idea: The change in ^{14}C is proportional to the existing amount.

$$\frac{d\,{}^{14}C}{dt} = k\,{}^{14}C.$$

So $\frac{d\,{}^{14}C}{{}^{14}C} = k\,dt$. So $\int \frac{d\,{}^{14}C}{{}^{14}C} = \int k\,dt$.

So $\ln {}^{14}C = kt + c$. So ${}^{14}C = e^{kt+c} = e^{kt}e^{c} = Ke^{kt}$,

where K is a constant.

Suppose that at $t=0$ the amount of ${}^{14}C$ is ${}^{14}C_0$.

Then $\quad {}^{14}C_0 = Ke^{k\cdot 0} = K$

So $\quad {}^{14}C = {}^{14}C_0 e^{kt}$.

The half life of ${}^{14}C$ is 5720 years. So,

at $t = 5720$

$$\tfrac{1}{2}\,{}^{14}C_0 = {}^{14}C_0 e^{kt} = {}^{14}C_0 e^{k5720}.$$

So $\quad \frac{1}{2} = e^{k5720}$. So $\ln(\frac{1}{2}) = k \cdot 5720$.

So $\quad k = \frac{\ln(\frac{1}{2})}{5720}$.

So $\quad {}^{14}C = {}^{14}C_0 e^{\frac{\ln(1/2)}{5720}t}$.

Now there is 54.2% of the original ${}^{14}C$. So

$(.542)\,{}^{14}C_0 = {}^{14}C_0 e^{\frac{\ln(1/2)}{5720}t}$. So $.542 = e^{\frac{\ln(1/2)}{5720}t}$.

So $\ln(.542) = \frac{\ln(\frac{1}{2})}{5720}t$. So $t = \frac{\ln(.542)\,5720}{\ln(1/2)}$.

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