MATH 221 Lecture 7 September 20, 2000

POOF! ... and there etched in stone was

$$e^{ix} = \cos x + i\sin x.$$

Do you believe it?

Example Verify $\cos(x+y) = \cos x\cos y - \sin x\sin y$

$\sin(x+y) = \sin x\cos y + \cos x\sin y.$

Well

$$e^{ix}e^{iy} = (\cos x + i\sin x)(\cos y + i\sin y)$$

$$= \cos x\cos y + i\cos x\sin y + i\sin x\cos y + i^2\sin x\sin y$$

$$= \cos x\cos y - \sin x\sin y + i(\sin x\cos y + \cos x\sin y)$$

On the other hand

$$e^{ix}e^{iy} = e^{i(x+y)} = \cos(x+y) + i\sin(x+y).$$

So

$\cos(x+y) = \cos x\cos y - \sin x\sin y$ and

$\sin(x+y) = \sin x\cos y + \cos x\sin y.$

Example Verify $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Well $e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$

$$= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$+ ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

On the other hand

$$e^{ix} = \cos x + i\sin x.$$

So $\cos x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$ and

$\sin x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right).$

Example Verify $\sin(-x) = -\sin x$ and

$\cos(-x) = \cos x.$

$$\sin(-x) = (-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \frac{(-x^7)}{7!} + \cdots$$

$$= -x - \frac{-x^3}{3!} + \frac{-x^5}{5!} - \frac{-x^7}{7!} + \cdots$$

$$= -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) = -\sin x$$

③

$$\cos(-x) = 1 - \frac{(-x)^2}{2!} + \frac{(-x)^4}{4!} - \frac{(-x)^6}{6!} + \frac{(-x)^8}{8!} - \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$= \cos x.$$

Example Verify $\sin^2 x + \cos^2 x = 1$.

Well
$$e^{ix}e^{-ix} = e^{ix-ix} = e^0 = 1.$$

On the other hand

$$e^{ix}e^{-ix} = (\cos x + i\sin x)(\cos(-x) + i\sin(-x))$$

$$= (\cos x + i\sin x)(\cos x - i\sin x)$$

$$= \cos^2 x - i\cos x\sin x + i\sin x\cos x - i^2\sin^2 x$$

$$= \cos^2 x + \sin^2 x.$$

So $\boxed{1 = \cos^2 x + \sin^2 x}$

Example Find $\frac{dy}{dx}$ when $y = \sin x$.

$$\frac{dy}{dx} = \frac{d\sin x}{dx} = \frac{d\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots\right)}{dx}$$

④

$$= 1 - \frac{3x^2}{3\cdot2\cdot1} + \frac{5x^4}{5\cdot4\cdot3\cdot2} - \frac{7x^6}{7\cdot6\cdot5\cdot4\cdot3\cdot2} + \cdots$$

$$= 1 - \frac{x^2}{2\cdot1} + \frac{x^4}{4\cdot3\cdot2} - \frac{x^6}{6\cdot5\cdot4\cdot3\cdot2} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \cos x.$$

So $\frac{d\sin x}{dx} = \cos x.$

Example Find $\frac{d\cos x}{dx}$.

$$\frac{d\cos x}{dx} = \frac{d\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right)}{dx}$$

$$= -\frac{2x}{2\cdot1} + \frac{4x^3}{4\cdot3\cdot2\cdot1} - \frac{6x^5}{6\cdot5\cdot4\cdot3\cdot2} + \frac{8x^7}{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2} - \cdots$$

$$= -x + \frac{x^3}{3\cdot2\cdot1} - \frac{x^5}{5\cdot4\cdot3\cdot2} + \frac{x^7}{7\cdot6\cdot5\cdot4\cdot3\cdot2} - \cdots$$

$$= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots$$

$$= -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= -\sin x.$$

So $\frac{d\cos x}{dx} = -\sin x.$

Inverse trig functions

The inverse function to f is the function that undoes f.

$\sin^{-1}x$ is the inverse function to $\sin x$

$\cos^{-1}x$ is the inverse function to $\cos x$

$\tan^{-1}x$ is the inverse function to $\tan x$

$\cot^{-1}x$ is the inverse function to $\cot x$

$\sec^{-1}x$ is the inverse function to $\sec x$

$\csc^{-1}x$ is the inverse function to $\csc x$.

So $\sin^{-1}(\sin B) = B$ and $\sin(\sin^{-1}x) = x$.

If $y = \sin x$ then $\sin^{-1}y = x$.

Since $0 = \sin 0$ then $\sin^{-1}0 = 0$

$1 = \sin\frac{\pi}{2}$ so $\sin^{-1}1 = \frac{\pi}{2}$

Warnings (1) $f(x) = \sin^{-1}x$ is not a function in the strictest sense.

$\sin^{-1}0 = 0$ or π or 2π or $-\pi$ or -2π or 10π or ...

This is similar to

$\sqrt{9}$, which is 3 or -3 depending on the context.

(2) $\cos^{-1}x \neq (\cos x)^{-1}$

$\cos^{-1}x$ is the function that undoes $\cos x$

$(\cos x)^{-1} = \frac{1}{\cos x}$.

For example:

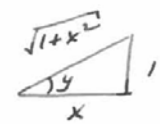
$(\cos 0)^{-1} = \frac{1}{\cos 0} = \frac{1}{1} = 1$. and

$\cos^{-1}0 = \frac{\pi}{2}$ or $\frac{-\pi}{2}$ or $\frac{3\pi}{2}$ or $\frac{-3\pi}{2}$ or ...

but $\cos^{-1}0$ is never equal to 1.

Example Verify $\sin(\cot^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

Let $y = \cot^{-1}x$. Then $\cot y = x$



So $\sin(\cot^{-1}x) = \sin y = \frac{1}{\sqrt{1+x^2}}$.

Example Verify $\sin^{-1}(-x) = -\sin^{-1}x$

$\sin^{-1}(-x) \stackrel{?}{=} -\sin^{-1}x$

$\sin(\sin^{-1}(-x)) \stackrel{?}{=} \sin(-\sin^{-1}x)$

$-x \stackrel{?}{=} -\sin(\sin^{-1}x)$

$-x \stackrel{?}{=} -x$ YES!!

Example Verify $\tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right)$.

Say $y = \tan^{-1}x$. Then $\tan y = x$.

So $\cot^{-1}\left(\frac{1}{x}\right) = \cot^{-1}\left(\frac{1}{\tan y}\right) = \cot^{-1}(\cot y) = y = \tan^{-1}x$.

Example Find $\frac{d\tan x}{dx}$.

$$\frac{d\tan x}{dx} = \frac{d\left(\frac{\sin x}{\cos x}\right)}{dx} = \frac{d(\sin x)(\cos x)^{-1}}{dx}$$

$$= \sin x\,\frac{d(\cos x)^{-1}}{dx} + \frac{d\sin x}{dx}(\cos x)^{-1}$$

$$= \sin x\,(-1)(\cos x)^{-2}\frac{d\cos x}{dx} + \cos x\,\frac{1}{\cos x}$$

$$= \frac{(-\sin x)(-\sin x)}{\cos^2 x} + 1 = \frac{+\sin^2 x + \cos^2 x}{\cos^2 x}$$

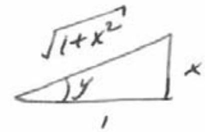
$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

Example Find $\frac{d\tan^{-1}x}{dx}$.

Set $y = \tan^{-1}x$. Then $\tan y = x$

Take the derivative: $\sec^2 y\,\frac{dy}{dx} = 1$.

So $\frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$



$$= \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}.$$

Example Find $\frac{d\sec x}{dx}$.

$$\frac{d\sec x}{dx} = \frac{d\left(\frac{1}{\cos x}\right)}{dx} = \frac{d(\cos x)^{-1}}{dx} = -(\cos x)^{-2}\frac{d\cos x}{dx}$$

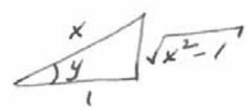
$$= -\frac{1}{\cos^2 x}(-\sin x) = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}$$

$$= \tan x\sec x.$$

Example Find $\frac{d\sec^{-1}x}{dx}$.

Let $y = \sec^{-1}x$. Then $\sec y = x$

So $\tan y\sec y\,\frac{dy}{dx} = 1$. So $\frac{dy}{dx} = \frac{1}{\tan y\sec y}$



So $\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}\;x}$

So $\frac{d\sec^{-1}x}{dx} = \frac{1}{x\sqrt{x^2-1}}$.